THE STRUCTURE OF THE CLASSIFYING RING OF FORMAL GROUPS WITH COMPLEX MULTIPLICATION.

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Abstract. If A is a commutative ring, there exists a classifying ring L^A of formal groups with complex multiplication by A, i.e., "formal A-modules." In this paper, the basic properties of the functor that sends A to L^A are developed and studied. When A is a Dedekind domain, the problem of computing L^A was studied by M. Lazard, by V. Drinfeld, and by M. Hazewinkel, who showed that L^A is a polynomial algebra whenever A is a discrete valuation ring or a (global) number ring of class number 1; Hazewinkel observed that L^A is not necessarily polynomial for more general Dedekind domains A, but no computations of L^A have ever appeared in any case when L^A is not a polynomial algebra. In the present paper, the ring L^A is computed, modulo torsion, for *all* Dedekind domains A of characteristic zero, including many cases in which L^A fails to be a polynomial algebra. Qualitative features (lifting and extensions) of the moduli theory of formal modules are then derived.

Contents

1. Introduction and review of some known facts.

1.1. **Introduction.** This paper[1] is about the classifying rings L^A and classifying Hopf algebroids (L^A, L^AB) of formal A-modules; or, from another point of view, the $fpqc$ moduli stack $\mathcal{M}_{fmA}$ of formal A-modules. I ought to explain what this means. When A is a commutative ring, a *formal A-module* is a formal group law F over an A-algebra R, which is

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[1]This paper is the first in a series about formal modules and homotopy theory, but the homotopy-theoretic aspects of formal modules do not have a significant presence until later papers in this series, such as [22].

additionally equipped with a ring map $\rho : A \to \mathrm{End}(F)$ such that $\rho(a)(X) \equiv aX$ modulo X^2. In other words, a formal A-module is a formal group law with complex multiplication by A. An excellent introductory reference for formal A-modules is [7]. Higher-dimensional formal modules exist, but *all formal modules in this paper will be implicitly assumed to be one-dimensional.*

Formal modules arise in algebraic and arithmetic geometry, for example, in Lubin and Tate's famous theorem (in [12]) on the abelian closure of a p-adic number field, in Drinfeld's generalizations of results of class field theory in [3], and in Drinfeld's p-adic symmetric domains, which are (rigid analytic) deformation spaces of certain formal modules; see [4] and [17]. In [23] I show that the moduli stack of formal A-modules has the property that its flat cohomology groups carry L-function data. Formal A-modules also arise in algebraic topology, by using the natural map from the moduli stack of formal A-modules to the moduli stack of formal groups to detect certain classes in the cohomology of the latter, particularly in order to resolve certain differentials in spectral sequences used to compute the Adams-Novikov E_2-term and stable homotopy groups of spheres; see e.g. [21] for these ideas.

More to the point for the present paper: it is easy to show (see [3]) that there exists a classifying ring L^A for formal A-modules, i.e., a commutative A-algebra L^A such that $\hom_{A-alg}(L^A, R)$ is in natural bijection with the set of formal A-modules over R. The hard part is actually computing this ring L^A:

- The pioneer in this area was M. Lazard, who, in the case $A = \mathbb{Z}$, proved (see [11]) that $L^{\mathbb{Z}} \cong \mathbb{Z}[x_1, x_2, \dots]$, a polynomial algebra on countably infinitely many generators. The ring $L^{\mathbb{Z}}$ is consequently often called the Lazard ring.
- Next, in [3], Drinfeld handled the case in which A is the ring of integers in a local nonarchimedean field (e.g. a p-adic number field). In that case Drinfeld proved that $L^A \cong A[x_1, x_2, \dots]$, again a polynomial algebra.
- In [7], Hazewinkel proved that the same result holds for discrete valuation rings, as well as for global number rings of class number one, that is, for all such rings A, the classifying ring L^A of formal A-modules is a polynomial A-algebra on countably infinitely many generators.
- Hazewinkel also makes the observation, in 21.3.3A of [7], that the same result cannot possibly hold for arbitrary global number rings. Specifically, when A is the ring of integers in $\mathbb{Q}(\sqrt[4]{-18})$, then Hazewinkel shows that the sub-A-module of L^A consisting of elements of grading degree 2 (see Theorem 1.2.1 for where this grading comes from) is not a free A-module, which could not occur if L^A were polynomial. Hazewinkel does not, however, attempt to compute L^A.

In fact, it seems that there are no computations of L^A in the literature whatsoever except in the cases that L^A turns out to be polynomial. This matters especially because there are qualitative features of formal A-modules which depend on whether or not L^A is polynomial. It was observed by Drinfeld, in [3], that, when A is the ring of integers in a nonarchimedean local field, then:

Extension: Every formal A-module n-bud extends to a formal A-module. (A formal module n-bud is a formal module $F(X, Y)$ only defined modulo $(X, Y)^{n+1}$ and which only is required to satisfy the axioms for a formal module modulo $(X, Y)^{n+1}$.)

Lifting: If R is a commutative A-algebra and I is an ideal of R, then every formal A-module over R/I is the modulo-I reduction of a formal A-module over R.

These two properties follow immediately from L^A being a polynomial A-algebra, and that is *how these properties are proven*: they do not follow from general results on formal A-modules, they follow from the explicit computation of L^A!

In the present paper, I compute L^A, modulo torsion, for all Dedekind domains A of characteristic zero. The specific result is Corollary 3.2.6: let A be a Dedekind domain of characteristic zero, let P denote the set of integers > 1 that are powers of prime numbers which are not invertible in A, and let R denote the set of integers > 1 not contained in P. Then we have an isomorphism of commutative graded A-algebras

$$Q_A(L^A) \cong A[x_{n-1} : n \in R] \otimes_A \bigotimes_A^{n \in P} \mathrm{Rees}_A^{2n-2}\left(I_n^A\right),$$

where the symbols are defined as follows:

- $Q_A(L^A)$ is the reduction of L^A modulo A-torsion,
- each polynomial generator x_{n-1} is in grading degree $2(n-1)$,
- $\mathrm{Rees}_A^{2n-2}(I_n^A)$ is the graded Rees A-algebra of I_n^A with I_n^A in grading degree $2n-2$,
- I_n^A is defined to be the ideal of A generated by $\nu(n)$ and all elements of A of the form $a - a^n$, and
- $\nu(n) = p$ if n is a power of a prime number p, and $\nu(n) = 1$ if n is not a prime power.

Consequently

$$Q_A(L^A) \cong A[x_{n-1} : n \in R] \otimes_A \bigotimes_A^{n \in P} \left(\bigotimes_A^{m \geq 1} A[y_{n^m}, z_{n^m}]/f_{n^m}(y_{n^m}, z_{n^m}) \right)$$

for some set of polynomials $\{f_{n^m}(y_{n^m}, z_{n^m}) : n \in P, m \geq 1\}$ (this is Corollary 3.2.7), with x_{n-1} in grading degree $2(n-1)$ and y_{n^m}, z_{n^m} each in grading degree $2(n^m - 1)$.

Consequently we get Corollary 3.2.8, a general analogue of Drinfeld's extension and lifting properties: if A is a Dedekind domain of characteristic zero, then:

Extension: Every formal A-module n-bud over a torsion-free commutative A-algebra R extends to a formal A-module over R.

Lifting: If R is a commutative A-algebra and I an ideal of R such that R/I is torsion-free, then every formal A-module over R/I is the reduction modulo I of a formal A-module over R.

As another consequence: when A is a Dedekind domain of characteristic zero with trivial class group, then L^A is a polynomial A-algebra, modulo torsion. On the other hand, when A is a Dedekind domain of characteristic zero with nontrivial class group, if any of the ideals I_n^A are nonprincipal, then L^A fails to be a polynomial algebra; however, L^A *modulo torsion is always a subalgebra of a polynomial A-algebra*.

The computation of L^A modulo torsion, and its various consequences, rely on having some basic properties of the functor $A \mapsto L^A$ already in place. Consequently in this paper I develop some of the basic properties of the functor that sends a commutative ring A to the commutative graded ring L^A (and also the functor that sends a commutative ring A to the graded Hopf algebroid (L^A, L^AB)). These basic properties are useful in their own right, and I make some use of them in the later papers in this series. Those properties are as follows:

Colimits: The functor sending A to L^A (and, more generally, sending A to the Hopf algebroid (L^A, L^AB)) commutes with filtered colimits and with coequalizers (but not, in general, coproducts). This is Proposition 2.2.1.

Localization: If A is a commutative ring and S a multiplicatively closed subset of A, then the homomorphism of graded rings $L^A[S^{-1}] \to L^{A[S^{-1}]}$ is an isomorphism. Furthermore, the homomorphism of graded Hopf algebroids

$$(L^A[S^{-1}], L^AB[S^{-1}]) \to (L^{A[S^{-1}]}, L^{A[S^{-1}]}B)$$

is an isomorphism. This is Theorem 2.3.3.

Localization and cohomology: If A is a commutative ring and S a multiplicatively closed subset of A, then for all graded left $L^A[S^{-1}]$-comodules M, we have an isomorphism

$$\left(\operatorname{Ext}^{s,t}_{(L^A,L^AB)}(L^A, M)\right)[S^{-1}] \cong \operatorname{Ext}^{s,t}_{(L^{A[S^{-1}]},L^{A[S^{-1}]}B)}(L^{A[S^{-1}]}, M)$$

for all nonnegative integers s and all integers t. This is Corollary 2.3.4.

Finite generation: The section 2.4 contains a variety of finiteness results: the ring L^A is a finitely-generated A-module in each grading degree if A is additively torsion-free and which is finitely generated as a ring, or a localization of such a ring, or if A is an additively torsion-free Henselian local ring of finite Krull dimension.

Completion: If A is an additively torsion-free commutative ring which is finitely generated as a ring, if I is a maximal ideal in A, and if A is separated in the I-adic topology, then the natural maps of commutative graded rings

$$L^A \otimes_A \hat{A}_I \to (L^A)\hat{}_I \to L^{\hat{A}_I}$$

are both isomorphisms. Even stronger, the maps of graded Hopf algebroids

$$(L^A \otimes_A \hat{A}_I, L^AB \otimes_A \hat{A}_I) \to ((L^A)\hat{}_I, (L^AB)\hat{}_I) \to (L^{\hat{A}_I}, L^{\hat{A}_I}B)$$

are both isomorphisms. This is Theorem 2.4.12.

Completion and cohomology: If A is an additively torsion-free commutative ring which is finitely generated as a ring, if I is a maximal ideal in A, if A is separated in the I-adic topology, if M is a graded left L^AB-comodule which is finitely generated as an A-module in each grading degree, and if M is bounded-below as a graded module, then, for all integers s,t with $s \geq 0$, we have isomorphisms of $\hat{A}_I$-modules

$$\operatorname{Ext}^{s,t}_{(L^A,L^AB)}(L^A, M) \otimes_A \hat{A}_I \cong \operatorname{Ext}^{s,t}_{(L^A,L^AB)}(L^A, \hat{M}_I) \cong \operatorname{Ext}^{s,t}_{(L^{\hat{A}_I},L^{\hat{A}_I}B)}(L^{\hat{A}_I}, \hat{M}_I)$$

for all integers s,t with $s \geq 0$. This is Corollary 2.4.13.

Base change and cohomology.: Let $A \to A'$ be a homomorphism of commutative rings, let N be a graded $L^{A'}B$-comodule which is flat as a $L^{A'}$-module, and let M be a graded right L^AB-comodule. Then we have an isomorphism

$$\operatorname{Ext}^{s,t}_{(L^A,L^AB)}(M, N) \cong \operatorname{Ext}^{s,t}_{(L^{A'},L^{A'}B)}(M \otimes_{L^A} L^{A'}, N)$$

for all nonnegative integers s and all integers t. This is Corollary 2.1.4.

These properties are not immediate consequences of the universal properties of L^A or of L^AB; they all require at least a little bit of work, usually a little bit of analysis (checking that certain sequences converge, and converge to unique limits) in the topological ring of endomorphisms of a formal group law.

In Proposition 2.1.3 I show that the classifying ring L^AB for formal A-modules is always a polynomial algebra over L^A, hence the Hopf algebroid (L^A, L^AB) is is isomorphic to $(L^A, L^A[b_1, b_2, \dots])$. The stack associated to the groupoid scheme $(\operatorname{Spec} L^A, \operatorname{Spec} L^AB)$ is the moduli stack of formal A-modules, so the reader who is so inclined can regard the computations in this paper as computations of presentations for this moduli stack. (It follows from Proposition 2.1.3 that the moduli stack of formal A-modules is a stack in the $fpqc$ topology, has affine diagonal, and admits a formally smooth cover by an affine scheme, namely Spec L^A, but this cover is only formally smooth and not smooth, since $L^A B$ is not finitely generated as an L^A-algebra; hence this moduli stack is "formally Artin" but not quite Artin.)

In section 1.2, I give a brief review of basic facts about Hopf algebroids and formal modules.

This paper is the first in a series; currently [24], [20], [22], and [23] are finished and available online. It is worth mentioning that in [20] I compute the classifying ring L^A explicitly for a large class of commutative rings A, including many Dedekind domains, but those results require that A be a localization of a finitely generated ring, so they do not render the ones in the present paper obsolete; see Remark 3.0.14.

I am grateful to D. Ravenel for teaching me a great deal about formal modules when I was a graduate student.

1.2. **Review of known facts about L^A and $L^A B$.** Nothing in this subsection is new, but I think it may be helpful to the reader have many of the basic ideas and known results on formal modules and their classifying Hopf algebroids collected in one place.

1.2.1. *Hopf algebroids.* This paper is about certain graded Hopf algebroids, i.e., cogroupoid objects in commutative graded rings. The graded-commutativity sign relation never occurs in this paper, so "commutative graded" really means "commutative and graded" and not "graded-commutative." The standard reference for Hopf algebroids is Appendix 1 of [19].

The results on classifying Hopf algebroids in this paper all have equivalent formulations in terms of moduli stacks. For concreteness, I have chosen to write the statements of results in terms of Hopf algebroids, and avoided writing them in terms of stacks. Readers familiar with and interested in algebraic stacks can easily use the cohomology-preserving correspondence between formally Artin stacks with affine diagonal and Hopf algebroids with formally smooth unit maps, as in e.g. [15], to rewrite this paper's results in terms of the flat cohomology of the moduli stack of formal A-modules.

I have also referred to "the moduli stack of formal A-modules" several times in this paper. This is slightly ambiguous for the following reason: formal A-modules as defined below in terms of power series, as a formal group law equipped with extra structure, actually have only a moduli prestack and not a moduli stack. This moduli prestack "stackifies" (see e.g. [10]) to a stack which is a moduli stack for "coordinate-free" formal A-modules, a situation which perfectly parallels that of formal group laws and formal groups, as in [25]. The details here are routine for the reader who is interested in stacks and formal algebraic geometry, and unimportant for the reader who is not.

1.2.2. *Formal modules.* If A is a commutative ring and R is a commutative A-algebra, then a (one-dimensional) *formal A-module over R* is a formal group law F over R, together with a ring homomorphism $\rho : A \to \mathrm{End}(F)$ such that the endomorphism $\rho(a) \in \mathrm{End}(F) \subseteq R[[X]]$ is congruent to aX modulo X^2. Here $\mathrm{End}(F)$ is a ring with addition given by formal addition (i.e., $f(X)$ added to $g(X)$ is $F(f(X), g(x))$, *not* the ordinary componentwise addition of power series), and with multiplication given by composition of power series (*not* the usual multiplication of power series in $R[[X]]$). Morally, F is a "formal group law with complex multiplication by A" (this perspective was taken already by Lubin and Tate in [12]). If n is a positive integer, a *formal A-module n-bud over R* is a formal group law n-bud over R, i.e., an element $F(X, Y) \in R[[X, Y]]/(X, Y)^{n+1}$ which satisfies the unitality, associativity, commutativity, and existence of inverses axioms modulo

$(X, Y)^{n+1}$, together with a ring homomorphism $\rho : A \to \mathrm{End}(F)$ such that the endomorphism $\rho(a) \in \mathrm{End}(F) \subseteq R[[X]]/(X^{n+1})$ is congruent to aX modulo X^2.

In this paper I will *always* write $\mathrm{End}(F)$ for the endomorphism ring of a formal group law F or formal group law n-bud F; even if F has the additional structure of a formal module, by $\mathrm{End}(F)$ I will mean the endomorphism ring of F as a formal group law or formal group law n-bud, without regard to any additional structure.

1.2.3. *The Hopf algebroid $(L^A, L^A B)$.* Theorem 1.2.1 is the main foundational result about the Hopf algebroid $(L^A, L^A B)$. It gathers together many results proven in chapter 21 of [7], although parts of the theorem are older than Hazewinkel's book; for example, the computation of the ring L^A, when A is a field or the ring of integers in a nonarchimedean local field, is due to Drinfeld in [3].

Theorem 1.2.1. *Let A be a commutative ring.*

- *Then there exist commutative A-algebras L^A and $L^A B$ having the following properties:*
  - *For any commutative A-algebra R, there exists a bijection, natural in R, between the set of A-algebra homomorphisms $L^A \to R$ and the set of formal A-modules over R.*
  - *For any commutative A-algebra R, there exists a bijection, natural in R, between the set of A-algebra homomorphisms $L^A B \to R$ and the set of strict isomorphisms of formal A-modules over R.*
- *The natural maps of sets between the set of formal A-modules over R and the set of strict isomorphisms of formal A-modules over R (sending a strict isomorphism to its domain or codomain, or sending a formal module to its identity strict isomorphism, or composing two strict isomorphisms, or sending a strict isomorphism to its inverse) are co-represented by maps of A-algebras between L^A and $L^A B$. Consequently $(L^A, L^A B)$ is a Hopf algebroid co-representing the functor sending a commutative A-algebra R to its groupoid of formal A-modules and their strict isomorphisms.*
- *If n is a positive integer, then the functor from commutative A-algebras to groupoids which sends a commutative A-algebra R to the groupoid of formal A-module n-buds over R and strict isomorphisms is also co-representable by a Hopf algebroid $(L^A_{\leq n}, L^A B_{\leq n})$, and since the groupoid of formal A-modules over R is the inverse limit over n of the groupoid of formal A-module n-buds over R, we have that*

  $$(L^A, L^A B) \cong \left(\mathrm{colim}_{n\to\infty} L^A_{\leq n}, \mathrm{colim}_{n\to\infty} L^A B_{\leq n}\right).$$

  For example, $L^A_{\leq 0} \cong A$ as commutative A-algebras. The filtration of L^A and $L^A B$ by $L^A_{\leq n}$ and $L^A B_{\leq n}$ induces a grading on L^A and on $L^A B$, in which the homogeneous grading degree $2n$ elements in L^A are the parameters for deforming (i.e., extending) a formal A-module n-bud to a formal A-module $(n+1)$-bud, and similarly, the homogeneous grading degree $2n$ elements in $L^A B$ are the parameters for deforming (i.e., extending) a formal A-module n-bud strict isomorphism to a formal A-module $(n+1)$-bud strict isomorphism. The summands of L^A and of $L^A B$ of odd grading degree are trivial.

- *If A is a field of characteristic zero or a discrete valuation ring or a global number ring of class number one, then we have isomorphisms of graded A-algebras*

$$\begin{aligned} L^A_{\leq n} &\cong A[x^A_1, x^A_2, x^A_3, \dots, x^A_n], \\ L^A B_{\leq n} &\cong L^A_{\leq n}[b^A_1, b^A_2, b^A_3, \dots, b^A_n], \quad \textit{and consequently} \\ L^A &\cong A[x^A_1, x^A_2, x^A_3, \dots], \\ L^A B &\cong L^A[b^A_1, b^A_2, b^A_3, \dots], \end{aligned}$$

with each x^A_i and each b^A_i homogeneous of grading degree $2i$. (However, the natural map $L^A \to L^B$ induced by a ring homomorphism $A \to B$ does not necessarily send each x^A_i to x^B_i!)

The factor of 2 in the gradings in Theorem 1.2.1 is due to the graded-commutativity sign convention in algebraic topology and the fact that $L^{\mathbb{Z}}$, with the above grading, is isomorphic to the graded ring of homotopy groups $\pi_*(MU)$ of the complex bordism spectrum MU, while $L^{\mathbb{Z}}B$ with the above grading is isomorphic to the graded ring $\pi_*(MU \wedge MU)$ of stable co-operations in complex bordism, and in fact $(L^{\mathbb{Z}}, L^{\mathbb{Z}}B) \cong (\pi_*(MU), \pi_*(MU \wedge MU))$ as graded Hopf algebroids. See [16] for these ideas.

Proposition 1.2.2 appears as Proposition 1.1 in [3].

Proposition 1.2.2. *Let A be a commutative ring, let n be an integer, and let L^A_n be the grading degree $2n$ summand in L^A. Let D^A_n be the sub-A-module of L^A generated by all products of the form xy where x, y are homogeneous elements of L^A of grading degree $< 2n$. If $n \geq 2$, then L^A_{n-1}/D^A_{n-1} is isomorphic to the A-module generated by symbols d and $\{c_a : a \in A\}$, that is, one generator c_a for each element a of A along with one additional generator d, modulo the relations:*

$$d(a - a^n) = \nu(n)c_a \ \textit{ for all } a \in A \tag{1.2.1}$$

$$c_{a+b} - c_a - c_b = d\frac{a^n + b^n - (a+b)^n}{\nu(n)} \ \textit{ for all } a, b \in A \tag{1.2.2}$$

$$ac_b + b^n c_a = c_{ab} \ \textit{ for all } a, b \in A. \tag{1.2.3}$$

I will call this Drinfeld's presentation for L^A_{n-1}/D^A_{n-1}.

The grading degrees in Proposition 1.2.2 are twice what they are in Drinfeld's statement of the result in [3]; this is to match the gradings that occur in algebraic topology, where the generator of $L^{\mathbb{Z}} \cong MU_*$ classifying an extension of a formal group n-bud to a formal group $(n+1)$-bud is in grading degree $2n$ rather than n.

Proposition 1.2.3 is an example of the usefulness of Drinfeld's presentation for L^A_{n-1}/D^A_{n-1}. Proposition 1.2.3 also appears as (part of) Proposition 21.2.10 in [7], but the proof given there goes only as far as to show that L^A_n/D^A_n is generated, as an A-module, by d, for each integer $n \geq 1$; the proof given in [7] claims, without explanation, that L^A_{n-1}/D^A_{n-1} is isomorphic to A and not some quotient A-module of A, and then concludes that, since each module of indecomposables L^A_{n-1}/D^A_{n-1} of L^A is isomorphic to A, we must have that $L^A \cong A[x_1, x_2, \dots]$, without explaining why L^A is not, for example, $A[x_1, x_2, \dots]/(x_1^2)$, or any other commutative graded A-algebra which is not polynomial but whose modules of indecomposables are all isomorphic to A. I prove these claims in the proof of Proposition 1.2.3. (The argument I use to prove these claims is not difficult, and must have been known to Hazewinkel, and it was not unreasonable for Hazewinkel to leave this part of the proof to be filled in by the interested reader.)

Proposition 1.2.3. *Let A be a commutative $\mathbb{Q}$-algebra. Suppose that A is additively torsion-free, i.e., if $m \in \mathbb{Z}$ and $a \in A$ and $ma = 0$, then either $m = 0$ or $a = 0$. Then the classifying ring L^A of formal A-modules is isomorphic, as a graded A-algebra, to $A[x_1, x_2, \dots]$, with x_n in grading degree $2n$, and with each x_n corresponding to the generator d of $A \cong L^A_n/D^A_n$ in the Drinfeld presentation for L^A_n/D^A_n.*

Proof. Since A is a $\mathbb{Q}$-algebra, we can solve the Drinfeld relation 1.2.1 to get

$$c_a = \frac{d(a - a^n)}{\nu(n)}$$

for all $a \in A$, and hence d generates L^A_{n-1}/D^A_{n-1} for all $n > 1$. We have the commutative graded A-algebra homomorphism $f : A \otimes_{\mathbb{Z}} L^{\mathbb{Z}} \to L^A$ classifying the underlying formal group law of the universal formal A-module, and since d generates L^A_n/D^A_n for all integers $n \geq 1$, the morphism f is an isomorphism on indecomposables. Hence f is surjective.

Now let F be the universal formal group law over $A \otimes_{\mathbb{Z}} L^{\mathbb{Z}}$. Since the universal formal group over $L^{\mathbb{Z}}$ has a logarithm, so does F (indeed, it is the same logarithm); write $\log_F$ for this logarithm. Then the ring map $\rho : A \to \mathrm{End}(F)$ given by $(\rho(a))(X) = \log_F^{-1}(a \log_F(X))$ makes F into a formal A-module. Let $g : L^A \to A \otimes_{\mathbb{Z}} L^{\mathbb{Z}}$ be the map classifying this formal A-module. Since the underlying formal group law of this formal A-module is again F, we get that $g \circ f = \mathrm{id}_{A \otimes_{\mathbb{Z}} L^{\mathbb{Z}}}$, hence f is injective. Since $L^{\mathbb{Z}} \cong \mathbb{Z}[x_1, x_2, \dots]$ by the classical work of Lazard, the claim in the statement of the proposition now follows. □

2. Generalities on L^A and $L^A B$.

2.1. Unique extension of complex multiplication along isomorphisms of formal groups. In this subsection I will freely use common notations for structure maps of bialgebroids and Hopf algebroids. The standard reference for bialgebroids and Hopf algebroids is Appendix 1 of [19]. In Proposition 2.1.1 I also refer to the Hopf algebroid Ext groups: if (A, Γ) is a commutative graded Hopf algebroid and M is a graded left Γ-comodule, I will write $\mathrm{Ext}^{s,t}_{(A,\Gamma)}(A, M)$ for the usual Ext-group, with s the cohomological degree and t the internal degree induced by the gradings on (A, Γ) and on M. Recall (from Appendix 1 of [19]) that the "usual" Ext-groups for categories of comodules over Hopf algebroids are the derived functors of $\hom(A, -)$, in the category of graded left Γ-comodules, *with respect to the allowable class generated by the comodules tensored up from* A; i.e., this is a relative Ext-group, in the sense of relative homological algebra, as in Chapter IX of [13]. These details do not arise again in this paper (although they become important in later papers in this series, e.g. [23], where I give computations of many of these Ext groups) and can be safely ignored by the reader whose interest in moduli of formal modules does not go as far as cohomology.

Proposition 2.1.1. *Let (R, Γ) be a commutative bialgebroid over a commutative ring A, and let S be a right Γ-comodule algebra, such that the following diagram commutes:*

(2.1.1)
$$\begin{tikzcd} R \arrow[r, "\eta_R"] \arrow[d, "f"] & \Gamma \arrow[d, "f \otimes_R \mathrm{id}_\Gamma"] \\ S \arrow[r, "\psi"] & S \otimes_R \Gamma \end{tikzcd}$$

where f is the R-algebra structure map $R \xrightarrow{f} S$. Then the algebraic object given by the pair $(S, S \otimes_R \Gamma)$, with its right unit $S \to S \otimes_R \Gamma$ equal to the comodule structure map ψ on

S, is a bialgebroid over A, and the map

$$(R,\Gamma) \to (S, S\otimes_R \Gamma), \tag{2.1.2}$$

with components f and ψ, is a morphism of bialgebroids.

If (R,Γ) is a Hopf algebroid, then so is $(S, S\otimes_R\Gamma)$, and 2.1.2 is a map of Hopf algebroids; if (R,Γ) is a graded bialgebroid and S is a graded right Γ-comodule algebra, then $(S, S\otimes_R \Gamma)$ is also a graded bialgebroid, and 2.1.2 is a map of graded bialgebroids; if (R,Γ) is a graded Hopf algebroid and S is a graded right Γ-comodule algebra, then $(S, S\otimes_R \Gamma)$ is also a graded Hopf algebroid, and 2.1.2 is a map of graded Hopf algebroids.

If, furthermore, the following conditions are also satisfied:

- *(R,Γ) is a graded Hopf algebroid which is connected (i.e., the grading degree zero summand Γ^0 of Γ is exactly the image of $\eta_L : R\to\Gamma$, equivalently $\eta_R : R\to\Gamma$), and*
- *S is a graded R-module concentrated in degree zero, and*
- *N is a graded left $S\otimes_R\Gamma$-comodule which is flat as an S-module, and*
- *M is a graded right Γ-comodule,*

then we have an isomorphism

$$\mathrm{Ext}^{s,t}_{(R,\Gamma)}(M,N) \cong \mathrm{Ext}^{s,t}_{(S,S\otimes_R\Gamma)}(M\otimes_R S, N)$$

for all nonnegative integers s and all integers t.

Proof. Note that the condition on the comodule algebra S guarantees that ψ "extends" η_R in a way that allows us to define a coproduct on $(S, S\otimes_R\Gamma)$, as we need an isomorphism $S\otimes_R\Gamma\otimes_R\Gamma \cong (S\otimes_R\Gamma)\otimes_S (S\otimes_R\Gamma)$.

First, we need to make explicit the structure maps on $(S, S\otimes_R\Gamma)$.T hroughout this proof, I will consistently use the symbols η_L, η_R, ϵ, and Δ (and χ if (R,Γ) is a Hopf algebroid) to denote the structure maps on (R,Γ), and $\psi : S\to S\otimes_R\Gamma$ to denote the comodule structure map of S. The augmentation, left unit, right unit, coproduct, and (if (R,Γ) is a Hopf algebroid) conjugation maps on $(S, S\otimes_R\Gamma)$ are, in order, the following maps:

$$\begin{aligned}
S\otimes_R\Gamma &\xrightarrow{\mathrm{id}_S\otimes_R\epsilon} S\\
S &\xrightarrow{\mathrm{id}_S\otimes_R\eta_L} S\otimes_R\Gamma\\
S &\xrightarrow{\psi} S\otimes_R\Gamma\\
S\otimes_R\Gamma &\xrightarrow{\mathrm{id}_S\otimes_R\Delta} S\otimes_R\Gamma\otimes_R\Gamma \cong (S\otimes_R\Gamma)\otimes_S (S\otimes_R\Gamma)\\
S\otimes_R\Gamma &\xrightarrow{\mathrm{id}_S\otimes_R\chi} S\otimes_R\Gamma.
\end{aligned}$$

We now need to show that these structure maps satisfy the axioms for being a bialgebroid. First we need to show that the coproduct on $(S, S\otimes_R\Gamma)$ is a left S-module morphism, i.e., that this diagram commutes:

$$\begin{tikzcd}
S \arrow[r, "\cong"] \arrow[d, "\mathrm{id}_S\otimes_R\eta_L"] & S\otimes_R R\otimes_R R \arrow[r, "\mathrm{id}_S\otimes_R\eta_L\otimes_R\eta_L"] & S\otimes_R\Gamma\otimes_R\Gamma \arrow[d, "\cong"] \\
S\otimes_R\Gamma \arrow[r, "\mathrm{id}_S\otimes_R\Delta"] & S\otimes_R\Gamma\otimes_R\Gamma \arrow[r, "\cong"] & (S\otimes_R\Gamma)\otimes_S (S\otimes_R\Gamma),
\end{tikzcd}$$

whose commutativity follows from Δ being a left R-module morphism.

We now check that the coproduct on $(S, S \otimes_R \Gamma)$ is also a right S-module morphism:

$$\begin{tikzcd} S \arrow[rr, "\cong"] \arrow[d, "\psi"] & & S \otimes_S S \arrow[d, "\psi \otimes_S \psi"] \\ S \otimes_R \Gamma \arrow[r, "\mathrm{id}_S \otimes_R \Delta"] & S \otimes_R \Gamma \otimes_R \Gamma \arrow[r, "\cong"] & (S \otimes_R \Gamma) \otimes_S (S \otimes_R \Gamma), \end{tikzcd}$$

whose commutativity follows from $(\mathrm{id}_S \otimes_R \Delta) \circ \psi = (\psi \otimes_R \mathrm{id}_\Gamma) \otimes_R \psi$, one of the axioms for S being a Γ-comodule.

We now check that the augmentation on $(S, S \otimes_R \Gamma)$ is a left S-module morphism, i.e., $\mathrm{id}_S = (\mathrm{id}_S \otimes_R \epsilon) \circ (\mathrm{id}_S \otimes_R \eta_L)$, which follows immediately from $\mathrm{id}_R = \epsilon \circ \eta_L$; and we check that the augmentation on $(S, S \otimes_R \Gamma)$ is a right S-module morphism, i.e., $(\mathrm{id}_S \otimes_R \epsilon) \circ \psi = \mathrm{id}_S$, which is precisely the other axiom for S being a Γ-comodule.

That the diagram

$$\begin{tikzcd} S \arrow[r, "\mathrm{id}_S \otimes_R \Delta"] \arrow[d, "\mathrm{id}_S \otimes_R \Delta"] & S \otimes_R \Gamma \otimes_R \Gamma \arrow[d, "\mathrm{id}_S \otimes_R \mathrm{id}_\Gamma \otimes_R \epsilon"] \\ S \otimes_R \Gamma \otimes_R \Gamma \arrow[r, "\mathrm{id}_S \otimes_R \epsilon \otimes_R \mathrm{id}_\Gamma"] & S \otimes_R \Gamma, \end{tikzcd}$$

commutes follows from the analogous property being satisfied by (R, Γ).

The last property we need to verify is the commutativity of the diagram:

$$\begin{tikzcd} S \otimes_R \Gamma \arrow[r, "\mathrm{id}_S \otimes_R \Delta"] \arrow[d, "\mathrm{id}_S \otimes_R \Delta"] & S \otimes_R \Gamma \otimes_R \Gamma \arrow[d, "\mathrm{id}_S \otimes_R \Delta \otimes_R \mathrm{id}_\Gamma"] \\ S \otimes_R \Gamma \otimes_R \Gamma \arrow[r, "\mathrm{id}_S \otimes_R \mathrm{id}_\Gamma \otimes_R \Delta"] & S \otimes_R \Gamma \otimes_R \Gamma \otimes_R \Gamma, \end{tikzcd}$$

which again follows immediately from the analogous property for (R, Γ).

In the graded cases, it is very easy to check by inspection of the above structure maps and diagrams that, since ψ is a graded map and all structure maps of (R, Γ) are graded maps, $(S, S \otimes_R \Gamma)$ and its structure maps are graded.

This proof has been put in terms of a right Γ-comodule algebra and $(S, S \otimes_R \Gamma)$ but the same methods work with obvious minor changes to give the stated result in terms of a left Γ-comodule algebra and $(S, \Gamma \otimes_R S)$.

The claims about Ext are a direct consequence of the standard Hopf algebroid change-of-rings isomorphism theorem, A1.3.12 in [19]. □

Proposition 2.1.2. *Let A be a commutative ring and let*

$$f : (R, \Gamma) \to (S, \Upsilon)$$

be a morphism of commutative Hopf algebroids over A. Write $f_{ob} : R \to S$ and $f_{mor} : \Gamma \to \Upsilon$ for the component maps of the morphism f of Hopf algebroids. Recall that, given a commutative A-algebra T, the set of A-algebra maps $R \to T$ is the set of objects of a natural groupoid $\hom_{A-alg}((R, \Gamma), T)$, and the set of A-algebra maps $\Gamma \to T$ is the set of morphisms of that same groupoid; and similarly for maps from S and Υ to T. Let f_T denote the morphism of groupoids

$$f_T : \hom_{A-alg}((S, \Upsilon), T) \to \hom_{A-alg}((R, \Gamma), T)$$

induced by f.

Then the two following conditions are equivalent:

- **(Isomorphism lifting condition.)** *For every object x in the groupoid $\hom_{A-alg}((S,\Upsilon),T)$ and every isomorphism $g : f_T(x) \xrightarrow{\cong} y$ in $\hom_{A-alg}((R,\Gamma),T)$, there exists a unique isomorphism $\tilde{g} : x \xrightarrow{\cong} \tilde{y}$ in $\hom_{A-alg}((S,\Upsilon),T)$ such that $f_T(\tilde{g}) = g$.*
- **(Base change condition.)** *S has the natural structure of a left Γ-comodule and there is an isomorphism of Hopf algebroids*

$$(S, S \otimes_R \Gamma) \xrightarrow{\cong} (S,\Upsilon)$$

making the diagram

$$\begin{tikzcd} (R,\Gamma) \arrow[r] \arrow[d] & (S,\Upsilon) \\ (S, S\otimes_R \Gamma) \arrow[ur, "\cong"] & \end{tikzcd}$$

commute, where the vertical map in the diagram is map 2.1.2 from Proposition 2.1.1.

Proof. Translating the isomorphism lifting condition into properties of maps out of R, Γ, S, and Υ, we get that the condition is equivalent to the claim that, for each commutative A-algebra T and each commutative square of A-algebra morphisms

$$\begin{tikzcd} R \arrow[r, "\eta_L"] \arrow[d, "f_{ob}"] & \Gamma \arrow[d, "\tau"] \\ S \arrow[r, "\sigma"] & T, \end{tikzcd}$$

there exists a unique A-algebra map $\upsilon : \Upsilon \to T$ making the diagram

$$\begin{tikzcd} R \arrow[r, "\eta_L"] \arrow[d, "f_{ob}"] & \Gamma \arrow[d, "f_{mor}"] \arrow[ddr, "\tau"] & \\ S \arrow[r, "\eta_L"] \arrow[drr, "\sigma"'] & \Upsilon \arrow[dr, "\upsilon"] & \\ & & T \end{tikzcd}$$

commute. In other words, Υ has exactly the universal property of the pushout, i.e., the tensor product $S \otimes_R \Gamma$, in the category of commutative A-algebras. □

Proposition 2.1.3. *Let $f : A \to A'$ be a homomorphism of commutative rings. Then the homomorphism of Hopf algebroids*

$$(L^A, L^A B) \to (L^{A'}, L^{A'} B) \tag{2.1.3}$$

classifying the underlying formal A-module of the universal formal A'-module and the underlying formal A-module strict isomorphism of the univeral formal A'-module strict isomorphism, satisfies the equivalent conditions of Proposition 2.1.2.

Proof. Suppose that F is a formal A'-module with action map $\rho_F : A' \to \mathrm{End}(F)$, and G is a formal A-module with action map $\rho_G : A \to \mathrm{End}(G)$. Suppose that $\phi(X)$ is a strict isomorphism of formal A-modules from the underlying formal A-module of F to G. Then

$$\phi\left((\rho_F \circ f)(a)(X)\right) = \rho_G(a)\left(\phi(X)\right)$$

for all $a \in A$, so if we let $\tilde{\rho}_G : A' \to \mathrm{End}(G)$ be defined by

$$\tilde{\rho}_G(a')(X) = \phi\left(\rho_F(a')(\phi^{-1}(X))\right), \tag{2.1.4}$$

for all $a' \in A'$, then

$$\phi\left(\rho_F(a')(X)\right) = \rho_G(a')\left(\phi(X)\right) \tag{2.1.5}$$

for all $a' \in A'$, i.e., ϕ is an isomorphism of formal A'-modules from F to G. Furthermore, applying ϕ^{-1} to 2.1.5 yields that 2.1.4 is the *only* formal A'-module structure on G making ϕ into an isomorphism of formal A'-modules. Hence the map 2.1.3 satisfies the isomorphism lifting condition of Proposition 2.1.2. □

The special case $M = L^A$ and $N = L^{A'}$ of Corollary 2.1.4, as well as the proof of Proposition 2.1.3, are not new: they also appear in [18] and [14].

Corollary 2.1.4. *Let $f : A \to A'$ be a homomorphism of commutative rings, let N be a graded $L^{A'}B$-comodule which is flat as a $L^{A'}$-module, and let M be a graded right L^AB-comodule. Then we have an isomorphism*

$$\mathrm{Ext}^{s,t}_{(L^A, L^AB)}(M, N) \cong \mathrm{Ext}^{s,t}_{(L^{A'}, L^{A'}B)}(M \otimes_{L^A} L^{A'}, N)$$

for all nonnegative integers s and all integers t.

2.2. **Colimits.**

Proposition 2.2.1. *Let $\mathcal{L}, \mathcal{LB}$ be the functors*

$$\begin{aligned} \mathcal{L} : \mathrm{Comm\,Rings} &\to \mathrm{Comm\,Rings} \\ \mathcal{L}(A) &= L^A \\ \mathcal{LB} : \mathrm{Comm\,Rings} &\to \mathrm{Comm\,Rings} \\ \mathcal{LB}(A) &= L^AB. \end{aligned}$$

Then $\mathcal{L}$ and $\mathcal{LB}$ each commute with filtered colimits, and $\mathcal{L}$ and $\mathcal{LB}$ each commute with coequalizers.

Proof. Let $\mathcal{D}$ be a small category. Suppose that either $\mathcal{D}$ is filtered or $\mathcal{D}$ is the category indexing a parallel pair, i.e., the Kronecker quiver

$$\bullet \rightrightarrows \bullet.$$

Let $G : \mathcal{D} \to \mathrm{Comm\,Rings}$ be a functor, let R be a commutative ring, and suppose we are given a cone $\mathcal{L} \circ G \to R$. Then R has the natural structure of a commutative $\mathrm{colim}\, G$-algebra, since the grading degree zero subring of each $\mathcal{L}(G(d))$ is isomorphic to the ring $G(d)$ itself. Since $\mathbb{Z}$ is initial in commutative rings, there is a unique cocone $\mathbb{Z} \to G$ and hence a canonical cocone $L^{\mathbb{Z}} \to \mathcal{L} \circ G$. Hence the cone $\mathcal{L} \circ G \to R$ describes a choice of formal group law F over the commutative $\mathrm{colim}\, G$-algebra R, together with a choice of ring map $\rho_d : G(d) \to \mathrm{End}(F)$ for each $d \in \mathrm{ob}\ \mathcal{D}$, compatible with the morphisms in $\mathcal{D}$, and such that $\rho_d(r)(X) \equiv rX$ modulo $(X^2) \subseteq (\mathrm{colim}\, G)[[X]]$ for all $r \in G(d)$. Since $\mathrm{End}(F)$ is typically not commutative, we need one small extra step before we can conclude that we get a ring map $\mathrm{colim}\, G \to \mathrm{End}(F)$, since the colimit $\mathrm{colim}\, G$ is computed in commutative rings, not in arbitrary associative rings: the image $\mathrm{im}\ \rho_d$ of each ρ_d is a commutative subring of $\mathrm{End}(F)$, so the union of the family of subrings $\cup_{d \in \mathrm{ob}\ \mathcal{D}} \mathrm{im}\ \rho_d$ is a commutative subring of $\mathrm{End}(F)$ since $\mathcal{D}$ is either filtered or is the category indexing parallel pairs. (This is the part that fails if $\mathcal{D}$ is an arbitrary small category; as far as I know there is no reason to believe that the conclusion of Proposition 2.2.1 holds for arbitrary small colimits, precisely because of the distinction between colimits in commutative rings and colimits in associative rings.) Hence we have a cone $G \to \cup_{d \in \mathrm{ob}\ \mathcal{D}} \mathrm{im}\ \rho_d$ in the category of commutative rings, hence a canonical map $\rho : \mathrm{colim}\, G \to \cup_{d \in \mathrm{ob}\ \mathcal{D}} \mathrm{im}\ \rho_d$ such that $\rho(r)(X) \equiv rX$ modulo X^2 for all $r \in \mathrm{colim}\, G$, hence F is a formal $\mathrm{colim}\, G$-module over R. Clearly if we began instead with a formal $\mathrm{colim}\, G$-module over R, by neglect of structure we get a cone $\mathcal{L} \circ G \to R$, and the two operations (sending such a cone to its $\mathrm{colim}\, G$-module, and sending the $\mathrm{colim}\, G$-module to its cone) are mutually inverse. So $\mathrm{colim}(\mathcal{L} \circ G) \cong \mathcal{L}(\mathrm{colim}\, G)$.

For $\mathcal{LB}$: we have already seen in Proposition 2.1.3 that $\mathcal{LB}$ is naturally equivalent to the functor $\mathcal{L} \otimes_{L^{\mathbb{Z}}} L^{\mathbb{Z}}B$. Since base change commutes with arbitrary colimits of commutative rings, the fact that $\mathcal{L}$ commutes with filtered colimits and coequalizers implies the same for $\mathcal{LB}$. □

Remark 2.2.2. Proposition 2.2.1 provides, at least in principle, a means of computing L^A and $L^A B$ for all commutative rings A: first, represent A as the coequalizer of a pair of maps

$$\mathbb{Z}[G] \leftleftarrows \mathbb{Z}[R] \tag{2.2.1}$$

where G is a set of generators and R a set of relations, and $\mathbb{Z}[G], \mathbb{Z}[R]$ are the free commutative algebras generated by the sets G and R, respectively. Then L^A is just the coequalizer, in commutative rings, of the two resulting maps $L^{\mathbb{Z}[R]} \to L^{\mathbb{Z}[G]}$.

Consequently, if one can compute L^A for polynomial rings A, then one can (at least in principle) compute L^A for all commutative rings A. Unfortunately, the computation of L^A for polynomial rings A is quite difficult (it does not seem to ever have been done for a polynomial ring on more than one indeterminate, and I do not do it in this paper, either), and since the functor $\mathcal{L}$ does not commute with coproducts, it is not as simple as computing $L^{\mathbb{Z}[x]}$ and then taking an n-fold tensor power to get $L^{\mathbb{Z}[x_1,\dots,x_n]}$, for example.

2.3. **Localization.** In the proof of Theorem 21.3.5 of Hazewinkel's excellent book [7], also appearing in the second edition [8], one finds the following statement:

> "By the very definition of L_A (as the solution of a certain universal problem) we have that $(L_A)_{\mathfrak{p}} = L_{A_{\mathfrak{p}}}$ for all prime ideals $\mathfrak{p}$ of A."

I find this statement mystifying: as far as I can tell, the universal properties of these rings do not imply that every formal A-module over a commutative $A_{\mathfrak{p}}$-algebra extends to a formal $A_{\mathfrak{p}}$-module, since the endomorphism ring $\mathrm{End}(F)$ of a formal group law defined over a ring R is typically not an R-algebra, as one sees from the famous example of the endomorphism ring of a height n formal group law over $\mathbb{F}_{p^n}$ being the maximal order in the invariant $1/n$ central division algebra over $\mathbb{Q}_p$, which is certainly not an $\mathbb{F}_{p^n}$-algebra. I have not been able to find any other proof of Hazewinkel's claim in the literature, either. Hazewinkel's claim also does not follow from Proposition 2.2.1, the fact that $A \mapsto L^A$ commutes with coequalizers and filtered colimits, since although localizations of *modules* can be defined as colimits in that category of modules, a localization of a *commutative ring* is not usually expressible as a colimit in the category of commutative rings (the morphisms in the diagram whose colimit computes the localization of the underlying module typically fail to be ring homomorphisms).

Nevertheless Hazewinkel is correct that the natural map of rings $L^A_{\mathfrak{p}} \to L^{A_{\mathfrak{p}}}$ is an isomorphism, and even better, the natural map of Hopf algebroids $(L^A_{\mathfrak{p}}, (L^A B)_{\mathfrak{p}}) \to (L^{A_{\mathfrak{p}}}, L^{A_{\mathfrak{p}}} B)$ is an isomorphism, although the proof is not quite as easy as an appeal to a universal property. In Proposition 2.3.3, I give the simplest proof that I have been able to find. This proof requires some preliminary lemmas. The first one is very elementary:

Lemma 2.3.1. *Let R be a commutative ring, and let $f(X) = r_1X+r_2X^2+r_3X^3+\dots \in R[[X]]$ be a power series over R with trivial constant term. If r_1 is a unit in R, then f admits a unique two-sided composition inverse, i.e, there exists a power series $\overline{f}(X) \in R[[X]]$, with trivial constant term and whose linear term is also a unit in R, such that $(\overline{f} \circ f)(X) = X = (f \circ \overline{f})(X)$.*

Proof. First, to find a left composition inverse for $f(X) = \sum_{i\geq 1} r_iX^i$, i.e., a power series $\overline{f}(X) = \sum_{i\geq 1} g_iX^i \in R[[X]]$ such that $(\overline{f} \circ f)(X) = X$, is equivalent to solving the system of equations

$$\begin{aligned} g_1r_1 &= 1, \\ g_1r_2 + g_2r_1^2 &= 0, \\ g_1r_3 + 2g_2r_1r_2 + g_3r_1^3 &= 0, \end{aligned}$$

and so on, arising from the equation

$$\begin{aligned} (\overline{f} \circ f)(X) &= X \\ &= g_1(r_1X + r_2X^2 + r_3X^3 + \dots) \\ &= +g_2(r_1X + r_2X^2 + r_3X^3 + \dots)^2 \\ &= +g_3(r_1X + r_2X^2 + r_3X^3 + \dots)^3 + \dots. \end{aligned}$$

After the first equation $g_1r_1 = 1$ (which is obviously solved uniquely by letting $g_1 = r_1^{-1}$), each equation in this system is of the form $g_nr_1^n + P_n(g_1, \dots, g_{n-1}, r_1, \dots, r_n) = 0$ for some polynomial P in $2n-1$ variables, and clearly if $g_1, \dots, g_{n-1}$ have already been uniquely solved for, then there is a unique solution g_n for this equation. Hence a unique left composition inverse $\overline{f}$ for f exists, and $\overline{f}$ has trivial constant term and its linear term is a unit in R. Hence $\overline{f}$ has its own left composition inverse $\overline{\overline{f}}$, and consequently

$$\overline{\overline{f}} = \overline{\overline{f}} \circ \overline{f} \circ f = f,$$

so $\overline{f}$ is both a left and a right composition inverse for f. □

Lemma 2.3.1 is certainly not new: in the case that R is a field of characteristic zero, the classical Lagrange inversion formula is the formula for the coefficients $g_1, g_2, \dots$ in the proof of Lemma 2.3.1!

Definition-Proposition 2.3.2. *Suppose R is a commutative ring. By $\mathcal{P}(R)$ I mean the monoid of power series $r_1X + r_2X^2 + r_3X^3 + \dots$ with trivial constant term and all coefficients in R, with the monoid operation in $\mathcal{P}(R)$ given by composition of power series.*

By a series composition ring over *R I mean an associative unital ring B equipped with an injective homomorphism of monoids $i : G(B) \to \mathcal{P}(R)$, where $G(B)$ is the underlying multiplicative monoid of B.*

Suppose that A is a commutative R-algebra, suppose that S is a multiplicatively closed subset of A containing 1, suppose that B is a series composition ring over $A[S^{-1}]$ with structure map $i : G(B) \to \mathcal{P}(A[S^{-1}])$, and suppose that we are given a ring homomorphism $f : A \to B$ such that $i(f(a)) \equiv aX$ modulo X^2 for all $a \in A$. Then there exists a unique ring homomorphism $\tilde{f} : A[S^{\pm 1}] \to B$ extending f. That ring homomorphism also has the property that $i(\tilde{f}(a/s)) \equiv (a/s)X$ modulo X^2 for all $a/s \in A[S^{-1}]$.

Proof. First, by the construction of the ring $A[S^{\pm 1}]$ and the fact that A is an integral domain, to specify a ring homomorphism $\tilde{f} : A[S^{\pm 1}] \to B$ extending f is the same thing as to specify, for each pair of elements (a, s) with $a \in A$ and $s \in S$, a choice of element $b_{r,s} \in B$, satisfying:

(1) $b_{a_1a_2,s_1s_2} = b_{a_1,s_1}b_{a_2,s_2}$ for all $a_1, a_2 \in A$ and all $s_1, s_2 \in S$,
(2) $b_{a_1s_2+a_2s_1,s_1s_2} = b_{a_1,s_1} + b_{a_2,s_2}$ for all $a_1, a_2 \in A$ and all $s_1, s_2 \in S$,
(3) $b_{0,s} = 0$ for all $s \in S$,
(4) $b_{s,s} = 1$ for all $s \in S$,
(5) $b_{a,1} = f(a)$ for all $a \in A$,
(6) if there exists some $u \in S$ such that $(a_1s_2 - s_1a_2)u = 0$, then $b_{a_1,s_1} = b_{a_2,s_2}$.

Then setting $\tilde{f}(a/s) = b_{a,s}$ yields the desired ring homomorphism. (Axiom 6 is only necessary when A is not an integral domain.)

Now we make some choices of elements $b_{a,s}$. For all $s \in S$, let $b_{1,s}$ be the (unique) composition inverse for $f(s)$, as in Lemma 2.3.1. Then, for all $a \in A$ and all $s \in S$, let $b_{a,s} = b_{1,s}f(a)$. Then clearly $b_{a,s}$ is congruent to $(a/s)X$ modulo X^2, as desired. We need to check that our choices of the elements $b_{a,s}$ satisfy the axioms 1, 2, 3, 4, 5, and 6. Axioms 3, 4, and 5 are automatic. Suppose that $a_1, a_2 \in A$ and $s_1, s_2 \in S$. Then we have equalities (using the fact that A is commutative, even though B is not necessarily commutative):

$$\begin{aligned} b_{a_1a_2,s_1s_2} &= b_{1,s_1s_2}f(s_1s_2)b_{a_1a_2,s_1s_2} \\ &= b_{1,s_1s_2}f(a_1a_2) \\ &= b_{1,s_1s_2}f(a_1)f(s_2)b_{1,s_2}f(a_2) \\ &= b_{1,s_1s_2}f(s_2)f(a_1)b_{1,s_2}f(a_2) \\ &= b_{1,s_1s_2}f(s_2)f(s_1)b_{1,s_1}f(a_1)b_{1,s_2}f(a_2) \\ &= b_{1,s_1s_2}f(s_1s_2)b_{1,s_1}f(a_1)b_{1,s_2}f(a_2) \\ &= b_{1,s_1}f(a_1)b_{1,s_2}f(a_2) \\ &= b_{a_1,s_1}b_{a_2,s_2}, \end{aligned}$$

which is precisely axiom 1, and

$$\begin{aligned} b_{a_1s_2+a_2s_1,s_1s_2} &= b_{1,s_1s_2}f(a_1s_2 + a_2s_1) \\ &= b_{1,s_1s_2}(f(s_2)f(s_1)b_{1,s_1}f(a_1) + f(s_1)f(s_2)b_{1,s_2}f(a_2)) \\ &= b_{1,s_1s_2}f(s_1s_2)(b_{1,s_1}f(a_1) + b_{1,s_2}f(a_2)) \\ &= b_{a_1,s_1} + b_{a_2,s_2}, \end{aligned}$$

which is precisely axiom 2. Hence we get a ring map $\tilde{f}(r/s) = b_{r,s}$ extending f.

Axiom 6 remains to be checked. Suppose that $u \in S$ and $(a_1s_2 - s_1a_2)u = 0$. Then we have equalities in $\mathcal{P}(A[S^{\pm 1}])$

$$\begin{aligned} b_{s_1s_2u,1}b_{a_1,s_1} &= b_{s_1a_1s_2u,s_1} \\ &= b_{1,s_1}f(s_1)f(a_1s_2u) \\ &= f(a_1s_2u) \\ &= f(s_1a_2u) \\ &= b_{1,s_2}f(s_2)f(s_1a_2u) \\ &= b_{s_2s_1a_2u,s_2} \\ &= b_{s_1s_2u,1}b_{a_2,s_2}, \end{aligned}$$

and since $s_1s_2u \in S$, the power series $b_{s_1s_2u,1}$ is invertible. Hence $b_{a_1,s_1} = b_{a_2,s_2}$, so axiom 6 is satisfied.

Note that any other choice of definition for $b_{1,s}$ would fail to simultaneously satisfy axioms 1, 4, and 5, and given our choices of $b_{1,s}$, any other choice of definition for $b_{r,s}$ for $r \neq 1$ would fail to simultaneously satisfy the same axioms. So these choices of $b_{r,s}$ are unique. So the extension $\tilde{f}$ of f is unique. $\square$

As far as I know, this notion of a "series composition ring" is new, but not of great importance: its only purpose is so that we can apply the result of Definition-Proposition 2.3.2 to the endomorphism ring $\mathrm{End}(F)$ of a formal group law F, which is the motivating example of a series composition ring.

Theorem 2.3.3. *Let A be a commutative ring and let S be a multiplicatively closed subset of A. Then the homomorphism of graded rings $L^A[S^{-1}] \rightarrow L^{A[S^{-1}]}$ is an isomorphism. Even better, the homomorphism of graded Hopf algebroids*

$$(L^A[S^{-1}], L^A B[S^{-1}]) \rightarrow (L^{A[S^{-1}]}, L^{A[S^{-1}]}B) \tag{2.3.1}$$

is an isomorphism of Hopf algebroids.

Proof. By Definition-Proposition 2.3.2, if F is a formal A-module defined over a commutative $A[S^{-1}]$-algebra, then the structure map $\rho_F : A \rightarrow \mathrm{End}(F)$ extends uniquely to a ring map $\tilde{\rho}_F : A[S^{-1}] \rightarrow \mathrm{End}(F)$ such that $\tilde{\rho}_F(a/s)(X) \equiv (a/s)X$ modulo X^2, i.e., F has the unique structure of a formal $A[S^{-1}]$-module extending its formal A-module structure. Hence the natural graded $A[S^{-1}]$-algebra map

$$L^A[S^{-1}] \rightarrow L^{A[S^{-1}]} \tag{2.3.2}$$

induces a natural bijection between the functors these two commutative $A[S^{-1}]$-algebra co-represent on the category of commutative $A[S^{-1}]$-algebras. Hence the map 2.3.2 is an isomorphism, by the Yoneda lemma.

Now using Proposition 2.1.3,

$$\begin{aligned} L^{A[S^{-1}]}B &\cong L^{A[S^{-1}]} \otimes_{L^A} L^A B \\ &\cong L^{A[S^{-1}]} \otimes_{L^A[S^{-1}]} (L^A B[S^{-1}]) \\ &\cong L^A B[S^{-1}], \end{aligned}$$

hence the map of Hopf algebroids 2.3.1 is an isomorphism. $\square$

Corollary 2.3.4. *Let A be a commutative ring and let S be a multiplicatively closed subset of A. Then, for all graded left $L^A[S^{-1}]$-comodules M, we have an isomorphism*

$$\left(\mathrm{Ext}^{s,t}_{(L^A, L^A B)}(L^A, M)\right)[S^{-1}] \cong \mathrm{Ext}^{s,t}_{(L^{A[S^{-1}]}, L^{A[S^{-1}]}B)}(L^{A[S^{-1}]}, M)$$

for all nonnegative integers s and all integers t.

In terms of the moduli stack $\mathcal{M}_{fmA}$ of formal A-modules:

Corollary 2.3.5. *Let A be a commutative ring and let S be a multiplicatively closed subset of A. Let f denote the stack homomorphism $f : \mathcal{M}_{fmA[S^{-1}]} \rightarrow \mathcal{M}_{fmA}$ classifying the underlying formal A-module of the universal formal $A[S^{-1}]$-module. Then, for all quasicoherent $\mathcal{O}_{\mathcal{M}_{fmA}}$-modules $\mathcal{F}$, we have an isomorphism*

$$H^n_{fl}(\mathcal{M}_{fmA}; \mathcal{F})[S^{-1}] \cong H^n_{fl}(\mathcal{M}_{fmA[S^{-1}]}; f^*\mathcal{F})$$

for all nonnegative integers s.

The additional grading on $H^*_{fl}(\mathcal{M}_{fmA})$ coming from the grading on the Hopf algebroid (L^A, L^AB) can be worked into the statement of Corollary 2.3.5 as an additional $\mathbb{G}_m$-action. The additional generality, however, complicates the statement considerably.

2.4. **Finiteness properties and completion.**

Lemma 2.4.1. *Let A be a commutative ring, and let R be a commutative graded A-algebra which is connective, i.e., the degree n grading summand R_n is trivial for all $n < 0$. Suppose that, for all integers n, the A-module R_n/D_n is finitely generated, where D_n is the sub-A-module of R_n generated by all elements of the form xy where x, y are homogeneous elements of R of grading degree $< n$.*

Then, for all integers n, R_n is a finitely generated A-module.

Proof. For each nonnegative integer n, choose a finite set of A-module generators $\{x_{n,i}\}_{i\in I_n}$ for R_n/D_n, where I_n is some finite indexing set. Then, for each nonnegative integer n and each $i \in I_n$, choose a lift $\overline{x}_{n,i}$ of $x_{n,i} \in R_n/D_n$ to R_n. Now the set consisting of all finitary products of elements $\{\overline{x}_{m,i} : m \leq n, i \in I_m\}$ is a set of A-module generators for R_n, and this set is finite. □

Proposition 2.4.2. *Let A be a commutative ring, and suppose that A is finitely generated as a commutative ring, i.e., there exists a surjective ring homomorphism $\mathbb{Z}[x_1, \dots, x_n] \to A$ for some positive integer n. Suppose further that A is additively torsion-free, i.e., the underlying abelian group of A is torsion-free.*

Then, for each integer m, the grading degree m summand L^A_m of the classifying ring L^A of formal A-modules is a finitely generated A-module.

Proof. Note that the "additively torsion-free" assumption implies that A is of characteristic zero. Suppose that A is generated, as a commutative ring, by a finite set of generators $x_1, \dots, x_n$. Then, to specify an A-module homomorphism from L^A_{m-1}/D^A_{m-1} to some other A-module M, it suffices to specify the images of d and $c_{x_1}, \dots, c_{x_n}$, since the images of the other elements c_a are all determined by the images of d and $c_{x_1}, \dots, c_{x_n}$ as well as the relations 1.2.2 and 1.2.3. (Relation 1.2.2 is the more important place where we use the assumption that A is additively torsion-free, so that there is at most one way of dividing $d(a^m + b^m - (a+b)^m)$ by $\nu(n) \in \mathbb{Z}$, so that c_{a+b} is *uniquely* determined by d and c_a and c_b.)

Consequently L^A_{m-1}/D^A_{m-1} is generated, as an A-module, by the $n+1$ elements $d, c_{x_1}, \dots, c_{x_n}$. Consequently L^A_{m-1}/D^A_{m-1} is a finitely generated A-module. (There are often relations in L^A_{m-1}/D^A_{m-1} between this set of $n+1$ generators, but that does not affect this proof.) Now Lemma 2.4.1 implies that L^A_m is a finitely generated A-module for all integers m. □

In Proposition 2.4.2 it is important that L^A is typically not a finitely-generated A-module, nor even finitely generated as an A-algebra; rather, the summand in each individual grading degree is a finitely generated A-module.

Corollary 2.4.3. *Let A be a commutative ring, and suppose that A is finitely generated as a commutative ring, and that A is additively torsion-free. Let I be a maximal ideal of A, and let A_I denote A localized at I, i.e., A with all elements outside of I inverted. Then, for each integer m, the grading degree m summand $L^{A_I}_m$ of the classifying ring L^{A_I} of formal A_I-modules is a finitely-generated, I-adically separated A_I-module.*

Proof. By Proposition 2.4.2, L^A_m is a finitely generated A-module for all integers m, and by Proposition 2.3.3, $(L^A)_I \cong L^{A_I}$, hence $(L^A_m)_I \cong L^{A_I}_m$ is a finitely generated A_I-module for all integers m. Since every finitely generated ring is Noetherian, A is Noetherian, hence A_I is

Noetherian and local and hence the Krull intersection theorem (see Corollary 10.20 in [1]) implies that every finitely generated A_I-module is I-adically separated. ◻

Proposition 2.4.4. *Let A be a Henselian local commutative ring with maximal ideal $\mathfrak{m}$, and suppose that A is torsion-free as an abelian group. Suppose that $\mathfrak{m}$ can be generated by κ elements, where κ is some cardinal number.*

Then, for each positive integer n, L^A_{n-1}/D^A_{n-1} can be generated, as an A-module, by:

- *$1+\kappa$ elements, if the residue field $A/\mathfrak{m}$ is isomorphic to a finite field $\mathbb{F}_q$ and n is a power of q,*
- *and 1 element (i.e., L^A_{n-1}/D^A_{n-1} is a cyclic A-module) otherwise.*

Proof. For this theorem we use Drinfeld's presentation for L^A_{n-1}/D^A_{n-1}, as in Proposition 1.2.2. Let p denote the characteristic of $A/\mathfrak{m}$ (so $p = 0$ is a possibility). There are three cases to consider:

- **If n is not a power of p:** Then $\nu(n)$ is not divisible by p, so $\nu(n) \in (A/\mathfrak{m})^\times$, so $\nu(n)$ is a unit in A since A is local. So we can solve relation 1.2.1 to get
$$c_a = \frac{d}{\nu(n)}(a - a^n)$$
for all $a \in A$. Hence L^A_{n-1}/D^A_{n-1} is generated by d.
- **If $n = p^t$ and either $A/\mathfrak{m}$ is infinite or $A/\mathfrak{m}$ has p^s elements and $s \nmid t$:** Then there exists some element $a \in A/\mathfrak{m}$ such that $a^{p^t} \neq a$, and since a is nonzero and A is Henselian, a lifts to an element $\tilde{a} \in A$ such that $\tilde{a}^{p^t} - \tilde{a}$ is not in the maximal ideal in A. Consequently $\tilde{a}^{p^t} - \tilde{a} \in A^\times$ and hence we can solve relation 1.2.1 to get
$$d = \frac{pc_{\tilde{a}}}{\tilde{a} - \tilde{a}^{p^t}}$$
and we can solve relation 1.2.3 to get
$$c_b = \frac{b - b^{p^t}}{\tilde{a} - \tilde{a}^{p^t}} c_{\tilde{a}},$$
hence L^A_{n-1}/D^A_{n-1} is generated by $c_{\tilde{a}}$.
- **If $n = p^t$ and $A/\mathfrak{m}$ has p^s elements and $s \mid t$:** This line of argument was inspired by Hazewinkel's Proposition 21.3.1 in [7]. Let M denote the A-submodule of L^A_{n-1}/D^A_{n-1} generated by all the elements c_m with $m \in \mathfrak{m}$. Solving relation 1.2.3, we get
$$(a - a^{p^t})c_m = (m - m^{p^t})c_a \tag{2.4.1}$$
for all $a, m \in A$. If $m \in \mathfrak{m}$, then $1 - m^{p^t-1} \notin \mathfrak{m}$, hence $1 - m^{p^t-1}$ is a unit since A is local. Hence
$$\frac{a - a^{p^t}}{1 - m^{p^t-1}} c_m = mc_a \tag{2.4.2}$$
for all $m \in \mathfrak{m}$ and all $a \in A$ with $a \notin \mathfrak{m}$. Furthermore
$$\frac{p}{1 - m^{p^t-1}} c_m = md, \tag{2.4.3}$$
and now equation 2.4.2 and 2.4.3 give us that $\mathfrak{m}$ acts by zero on $(L^A_{n-1}/D^A_{n-1})/M$, i.e., $(L^A_{n-1}/D^A_{n-1})/M$ is an $A/\mathfrak{m}$-vector space.

  Now s divides t, and hence $x^{p^t} = x$ for all $x \in A/\mathfrak{m}$, so relation 1.2.2 becomes $c_{a+b} = c_a + c_b$ in $(L^A_{n-1}/D^A_{n-1})/M$, and relation 1.2.3 becomes $c_{ab} = ac^b + bc^a$, i.e., the map $c : A/\mathfrak{m} \to (L^A_{n-1}/D^A_{n-1})/M$ is a $\mathbb{Z}$-module derivation. But the relevant module of Kähler differentials vanishes, $\Omega^1_{(A/\mathfrak{m})} \cong 0$, since $A/\mathfrak{m}$ is a field, so c factors through the zero module, so c is the zero map. So $c_a = 0$ in $(L^A_{n-1}/D^A_{n-1})/M$ for all $a \in A$ with $a \notin \mathfrak{m}$, and $c_a = 0$ in $(L^A_{n-1}/D^A_{n-1})/M$ for all $a \in \mathfrak{m}$ by the definition of M. So $(L^A_{n-1}/D^A_{n-1})/M \cong A/\mathfrak{m}$ generated by d, and the elements c_m for $m \in \mathfrak{m}$, along with d, form a set of generators for L^A_{n-1}/D^A_{n-1}.

□

Corollary 2.4.5. *Let A be a Henselian local commutative ring with maximal ideal $\mathfrak{m}$, and suppose that A is torsion-free as an abelian group and that $\mathfrak{m}$ is finitely generated. Then, for each positive integer n, L^A_{n-1}/D^A_{n-1}, is a finitely generated A-module.*

Corollary 2.4.6. *Let A be a Noetherian complete local commutative ring whose maximal ideal $\mathfrak{m}$ is finitely generated, and suppose that A is torsion-free as an abelian group. Then, for each integer n, the grading degree n summand L^A_n of the classifying ring L^A of formal A-modules is $\mathfrak{m}$-adically separated and $\mathfrak{m}$-adically complete.*

Proof. Corollary 2.4.5 tells us that L^A_n is a finitely generated A-module. Krull's intersection theorem (see Corollary 10.20 in [1]) implies that every finitely generated A-module is $\mathfrak{m}$-adically separated, and it is elementary (but I still provide a proof) that every finitely generated module over a Noetherian complete local ring with maximal ideal $\mathfrak{m}$ is $\mathfrak{m}$-adically complete: if M is finitely generated, it is also finitely presented since A is Noetherian, so there exists an exact sequence of A-modules

$$\coprod_{j\in J} A \to \coprod_{i\in I} A \to M \to 0$$

with I, J finite sets; hence, applying the $\mathfrak{m}$-adic completion functor (which is exact on finitely generated modules over a Noetherian ring; see Proposition 10.12 in [1]), we get the commutative diagram with exact rows

$$\begin{tikzcd}
\coprod_{j\in J} A \arrow[r] \arrow[d, "\cong"] & \coprod_{i\in I} A \arrow[r] \arrow[d, "\cong"] & M \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d, "\cong"] & 0 \arrow[d, "\cong"] \\
\left(\coprod_{j\in J} A\right)^{\hat{}}_{\mathfrak{m}} \arrow[r] & (\coprod_{i\in I} A)^{\hat{}}_{\mathfrak{m}} \arrow[r] & \hat{M}_{\mathfrak{m}} \arrow[r] & \hat{0}_{\mathfrak{m}} \arrow[r] & \hat{0}_{\mathfrak{m}},
\end{tikzcd}$$

and now the Five Lemma gives us that M is $\mathfrak{m}$-adically complete. □

Lemma 2.4.7. *Let R be a $\mathbb{Z}$-graded commutative ring which is connective, i.e., there exists some integer n such that $R^m \cong 0$ for all $m < n$; furthermore, assume that R^0 is Noetherian and that R^i is a finitely generated R^0-module for each integer i.*

Then, for any $\mathbb{Z}$-graded finitely generated R-module M and any ideal I of R generated by elements in R^0, the natural map

$$\hat{R}_I \otimes_R M \to \hat{M}_I$$

is an isomorphism of $\mathbb{Z}$-graded $\hat{R}_I$-modules.

(This is immediate, when R is Noetherian; the use of this lemma is when R is not Noetherian but R^0 is, e.g. $R \cong MU_* \cong L^{\mathbb{Z}}$.)

Proof. Since M is finitely generated as an R-module, M^i is finitely generated as an R^0-module for any integer i, and since M^i is a finitely generated module over a Noetherian ring, the map $\hat{R}^0_I \otimes_{R^0} M^i \to \hat{M}^i_I$ is an isomorphism (see e.g. Proposition 10.13 of [1]) for all i. □

Definition-Proposition 2.4.8. *Let A be a commutative ring and let I be an ideal in A. Let F be a formal group law defined over an A-algebra R. By the* canonical I-ideal in $\mathrm{End}(F)$, *which I will abbreviate $\mathfrak{c}$, I mean the two-sided ideal*

$$\mathfrak{c} = \{if(X) + Xg(X) : f(X), g(X) \in \mathrm{End}(F), i \in I\} \subseteq \mathrm{End}(F) \subseteq R[[X]]$$

of $\mathrm{End}(F)$.

Proof. I need to show that $\mathfrak{c}$ is a two-sided ideal. Clearly $\mathfrak{c}$ is closed under addition. Suppose that $q(X) \in \mathrm{End}(F)$. Then $if(q(X)) + q(X)g(q(X))$ is in $\mathfrak{c}$ since every endomorphism of a formal group law has trivial constant term, so $X \mid q(X)$. Furthermore, if $q(X) = \sum_{n=1}^{\infty} q_n X^n$ where $q_1, q_2, \dots \in R$, then

$$q(if(X) + Xg(X)) = \sum_{n=1}^{\infty} q_n (if(X) + Xg(X))^n$$

is in $\mathfrak{c}$, as one sees by using the binomial theorem to expand the powers and see that each monomial is divisible by either i or X. (This proof is very elementary but I included the details to try to avoid any confusion about the fact that the multiplication in $\mathrm{End}(F)$ is composition and not multiplication of power series.) ◻

Lemma 2.4.9. *Let A be a Noetherian commutative ring and let I be an ideal in A. Let R be a commutative A-algebra, and let F be a formal A-module. If R is I-adically separated, then* $\mathrm{End}(F)$ *is $\mathfrak{c}$-adically separated. If R is I-adically complete, then* $\mathrm{End}(F)$ *is $\mathfrak{c}$-adically complete.*

Proof. Suppose that R is I-adically separated, and suppose that $\zeta(X) \in \cap_{n\in\mathbb{N}} \mathfrak{c}^n \subseteq \mathrm{End}(F)$. Then $\zeta(X) \in \mathfrak{c}^n$ for all positive integers n, so in particular, for all positive integers n,

$$\zeta(X) = i^n f_n(X) + X^n g_n(X) \tag{2.4.4}$$

for some $i \in IR$ and some $f_n(X), g_n(X) \in \mathrm{End}(F)$. If we write $\zeta(X)$ as $\zeta(X) = \sum_{j=1}^{\infty} \zeta_j X^j$, then for each positive integer j, equation 2.4.4 in the case $n > j$ implies that $\zeta_j \in I^n R$. Since this is true for all $n > j$, this implies that $\zeta_j \in \cap_{n\in\mathbb{N}} I^n R = \{0\}$, hence $\zeta(X) = 0$, and hence $\mathrm{End}(F)$ is $\mathfrak{c}$-adically separated.

Now suppose instead that R is I-adically complete, and that

$$(\zeta_1(X), \zeta_2(X), \dots) \tag{2.4.5}$$

is a Cauchy sequence in the $\mathfrak{c}$-adic topology on $\mathrm{End}(F)$. For each positive integer i, write $\zeta_i(X)$ as

$$\zeta_i(X) = \sum_{k=1}^{\infty} \zeta_{i,k} X^k$$

for some $\zeta_{i,1}, \zeta_{i,2}, \dots \in R$. Then, by the Cauchy condition, for each positive integer n there exists a positive integer m such that the difference between $\zeta_{i,k}$ and $\zeta_{j,k}$ is in $I^n R$ as long as $k < n$ and $i, j \geq m$. Consequently, for each k, the sequence of elements $(\zeta_{1,k}, \zeta_{2,k}, \dots)$ of R is I-adically Cauchy. Since R is assumed I-adically complete, $(\zeta_{1,k}, \zeta_{2,k}, \dots)$ converges to some element, which I will call $\hat{\zeta}_k$. Then the power series $\sum_{k=1}^{\infty} \hat{\zeta}_k X^k$ is easily seen to be the limit of the sequence 2.4.5. So every $\mathfrak{c}$-adic Cauchy sequence in $\mathrm{End}(F)$ converges, so $\mathrm{End}(F)$ is $\mathfrak{c}$-adically complete. ◻

In Lemma 2.4.10 and in Theorem 2.4.12 there is an assumption that A is a Noetherian commutative ring with an ideal I such that A is I-adically separated, i.e., $\cap_{n\in\mathbb{N}} I^n = \{0\} \subseteq A$. It may be helpful to remind the reader that this condition is very commonly satisfied, due to Krull's intersection theorem (see e.g. Corollary 10.19 in [1]), which has the consequence that if I is contained in the Jacobson radical of A, then A is I-adically separated. An even more straightforward consequence of Krull's theorem is that, if A is an integral domain, then with no assumptions on I, we get that A is I-adically separated. Both of these consequences of Krull's theorem do use the assumption that A is Noetherian.

Lemma 2.4.10. *Let A be a Noetherian commutative ring and let I be an ideal in A. Suppose that A is separated (but not necessarily complete) in the I-adic topology. Let R be a commutative A-algebra which is I-adically separated and complete, and let F be a formal A-module over R. Then F is the underlying formal A-module of exactly one formal $\hat{A}_I$-module. That is, the action map $\rho : A \to \mathrm{End}(F)$ extends uniquely to an action map $\tilde{\rho} : \hat{A}_I \to \mathrm{End}(F)$ making F a formal $\hat{A}_I$-module.*

Proof. Choose an element $a \in \hat{A}_I$, and for each positive integer n, let a_n be the image of a under the projection map $\hat{A}_I \to A/I^n$, and let $\tilde{a}_n$ be an element of A whose reduction modulo I^n is a_n. (In other words: choose a sequence of elements $(\tilde{a}_1, \tilde{a}_2, \dots)$ of A converging to a in the I-adic topology.) Then the sequence $(\tilde{a}_1, \tilde{a}_2, \dots)$ uniquely determines the element $a \in \hat{A}_I$, since we assumed that A is separated in the I-adic topology.

The tangency condition on ρ (that $\rho(X) \equiv X \mod X^2$) implies that the image of I under ρ is contained in the canonical I-ideal $\mathfrak{c}$ of Definition-Proposition 2.4.8. By Lemma 2.4.9, $\mathrm{End}(F)$ is $\mathfrak{c}$-adically separated and complete, so the sequence $(\rho(\tilde{a}_1), \rho(\tilde{a}_2), \dots)$ in $\mathrm{End}(F)$ converges to a unique element in $\mathrm{End}(F)$. Let $\tilde{\rho}(a)$ be defined to be this element. It is elementary to check that the resulting map $\tilde{\rho} : \hat{A}_I \to \mathrm{End}(F)$ is a well-defined ring homomorphism and agrees with ρ when composed with the injection $A \hookrightarrow \hat{A}_I$. (This map $\tilde{\rho}$ is, of course, the one given by the universal property of completion, but I am giving some detail here because $\mathrm{End}(F)$ is not typically commutative and ρ does not typically have its image inside the center of $\mathrm{End}(F)$, so the situation is not exactly the textbook one.) The tangency condition for $\tilde{\rho}$ is similarly easy: any element $a \in \hat{A}_I$ can be approximated arbitrarily $\mathfrak{c}$-adically closely by an element of A, and $\tilde{\rho}$ satisfies the tangency condition on elements of A since $\tilde{\rho}$ coincides with ρ on elements of A.

Consequently F is indeed the underlying formal A-module of a formal $\hat{A}_I$-module. The fact that the ring homomorphism $\tilde{\rho}$ is the *unique* extension of ρ to a ring map $\hat{A}_I \to \mathrm{End}(F)$ is as follows: any ring homomorphism $\hat{A}_I \to \mathrm{End}(F)$ extending ρ sends I into $\mathfrak{c}$ and hence is continuous, hence is a continuous homomorphism of abelian groups; now the universal property of the completion implies that the extension $\tilde{\rho}$ is unique. □

Lemma 2.4.11. *Let A be a Noetherian commutative ring and let I be an ideal in A. Let R be a commutative $\hat{A}_I$-algebra, and let F, G be formal $\hat{A}_I$-modules over R. Suppose that $f : F \to G$ is a strict isomorphism of the underlying formal A-modules of F and G. Then f is also a strict isomorphism $F \to G$ of formal $\hat{A}_I$-modules.*

Proof. Let $\rho_F : \hat{A}_I \to \mathrm{End}(F)$ and $\rho_G : \hat{A}_I \to \mathrm{End}(G)$ denote the structure maps of F and G as formal $\hat{A}_I$-modules, respectively. Then $\rho_F(f(a)(X)) = f(\rho_G(a)(X))$ for all $a \in A$, and we need to show that the same is true for all $a \in \hat{A}_I$. For any $a \in \hat{A}_I$, choose a sequence of elements $a_1, a_2, \dots$ in A such that $\lim_{n\to\infty} a_n = a$ in the I-adic topology. Then the fact that ρ_F and ρ_G are continuous (since each sends I into $\mathfrak{c}$) implies that

$$f(\rho_F(\lim_{n\to\infty} a_n)(X)) = f(\lim_{n\to\infty} \rho_F(a_n)(X)), \text{ and} \tag{2.4.6}$$

$$\rho_G(\lim_{n\to\infty} a_n)(f(X)) = \lim_{n\to\infty} \rho_G(a_n)(f(X)). \tag{2.4.7}$$

Now since f is a homomorphism of formal group laws, its constant coefficient is zero, and hence f is continuous in a limited sense: whenever $\xi_1(X), \xi_2(X), \dots$ is a $\mathfrak{c}$-adically convergent sequence in $\mathrm{End}(F)$ such that each power series $f(\xi_1(X)), f(\xi_2(X)), \dots$ is contained in $\mathrm{End}(G) \subseteq R[[X]]$ and $\mathfrak{c}$-adically convergent, we get an equality $\lim_{n\to\infty} f(\xi_n(X)) = f(\lim_{n\to\infty} \xi_n(X))$. Consequently 2.4.6 is equal to 2.4.7, and hence $\rho_F(f(a)(X)) = f(\rho_G(a)(X))$. □

Theorem 2.4.12. *Let A be a commutative ring which is finitely generated as a commutative ring,and suppose that the underlying abelian group of A is torsion-free. Let I be a maximal ideal of A, and suppose that A is separated in the I-adic topology. (This last condition is automatic if A is also an integral domain, by Krull's intersection theorem.) Then the natural maps of graded Hopf algebroids*

$$(L^A \otimes_A \hat{A}_I, L^A B \otimes_A \hat{A}_I) \to ((L^A)\hat{}_I, (L^A B)\hat{}_I) \tag{2.4.8}$$

$$\to (L^{\hat{A}_I}, L^{\hat{A}_I} B) \tag{2.4.9}$$

are isomorphisms. (This is stronger than just being an equivalence. In particular, the natural maps $L^A \otimes_A \hat{A}_I \to (L^A)\hat{}_I \to L^{\hat{A}_I}$ and $L^A B \otimes_A \hat{A}_I \to (L^A B)\hat{}_I \to L^{\hat{A}_I} B$ are all isomorphisms of graded rings.)

Proof. By Proposition 2.4.2, for all integers n the grading degree n summand $(L^A)^n$ in the ring L^A is a finitely-generated A-module, and $L^A B \cong L^A B \otimes_L LB \cong L^A[b_1, b_2, \dots]$ (by Proposition 2.1.3) is also a finitely-generated A-module in each grading degree (see e.g. 2.14 of [14] for the isomorphism $L^A B \cong L^A[b_1, b_2, \dots]$ with the b_i generators in distinct positive grading degrees). Consequently Lemma 2.4.7 applies, since A is finitely generated as a commutative ring and hence Noetherian, even though L^A is typically not Noetherian; so the map 2.4.8 is an isomorphism.

The more substantial result is that 2.4.9 is also an isomorphism. By Lemma 2.4.10, Lemma 2.4.11, and the universal properties of the rings involved, the map 2.4.9 induces bijections

$$\hom_{\hat{A}_I-alg}(L^{\hat{A}_I}, R) \xrightarrow{\cong} \hom_{\hat{A}_I-alg}(L^A \otimes_A \hat{A}_I, R) \text{ and}$$
$$\hom_{\hat{A}_I-alg}(L^{\hat{A}_I} B, R) \xrightarrow{\cong} \hom_{\hat{A}_I-alg}(L^A B \otimes_A \hat{A}_I, R),$$

natural in R, for all commutative $\hat{A}_I$-algebras R which are I-adically separated and complete.

Now the Yoneda lemma tells us that the ring maps $L^A \otimes_A \hat{A}_I \to L^{\hat{A}_I}$ and $L^A B \otimes_A \hat{A}_I \to L^{\hat{A}_I} B$ are isomorphisms, as long as all four of these rings are actually objects in the category of I-adically separated and complete commutative $\hat{A}_I$-algebras! Now L^{A_I} a finitely generated A-module in each grading degree by Corollary 2.4.3, hence the same is true of $(L^{A_I})\hat{}_I \cong L^{A_I} \otimes_{A_I} \hat{A}_I$, hence $(L^{A_I})\hat{}_I$ is I-adically separated and I-adically complete by the same argument as in the proof of Corollary 2.4.6, and the same is true for $(L^{A_I} B)\hat{}_I \cong (L^{A_I})\hat{}_I \otimes_L LB$, by Proposition 2.1.3. On the other hand, $L^{\hat{A}_I}$ is I-adically separated and complete in each grading degree by Corollary 2.4.6, hence $L^{\hat{A}_I} B \cong L^{\hat{A}_I} \otimes_L LB$ is as well, again by Proposition 2.1.3. □

Corollary 2.4.13. *Let A be a commutative ring which is finitely generated as a commutative ring, and suppose that A is additively torsion-free. Let I be a maximal ideal of A, and suppose that A is separated in the I-adic topology. (This last condition is automatic if A is also an integral domain, by Krull's intersection theorem.) Let M be a graded left $L^A B$-comodule which is finitely-generated as an A-module in each grading degree, and* *suppose that M is bounded-below, i.e., there exists some integer b such that M is trivial below grading degree b. (For example, $M = L^A$ satisfies all these conditions on M.) Then, for all integers s, t with $s \geq 0$, we have isomorphisms of $\hat{A}_I$-modules*

$$\mathrm{Ext}^{s,t}_{(L^A, L^A B)}(L^A, M) \otimes_A \hat{A}_I \cong \mathrm{Ext}^{s,t}_{(L^A, L^A B)}(L^A, \hat{M}_I) \tag{2.4.10}$$

$$\cong \mathrm{Ext}^{s,t}_{(L^{\hat{A}_I}, L^{\hat{A}_I} B)}(L^{\hat{A}_I}, \hat{M}_I) \tag{2.4.11}$$

Proof. Let $C^\bullet_{(L^A, L^A B)}(M)$ be the cobar complex of the Hopf algebroid $(L^A, L^A B)$ with coefficients in M. (See Appendix 1 of [19] for the definition and basic properties of the cobar complex; most pertinent for us right now is that its module of n-cochains $C^n_{(L^A, L^A B)}(M)$ is isomorphic to $(L^A B)^{\otimes_{L^A} n} \otimes_{L^A} M$, and the cohomology of $C^\bullet_{(L^A, L^A B)}(M)$ is $\mathrm{Ext}^{*,*}_{(L^A, L^A B)}(L^A, M)$.) Then, since $L^A B$ is a finitely generated A-module in each grading degree by Proposition 2.4.2, the same is true of $(L^A B)^{\otimes_{L^A} n} \otimes_{L^A} M$. Consequently we have isomorphisms

$$\begin{aligned}
(L^A B)^{\otimes_{L^A} n} \otimes_{L^A} \hat{M}_I &\cong (L^A B)^{\otimes_{L^A} n} \otimes_{L^A} M \otimes_A \hat{A}_I \\
&\cong \left(L^A B \otimes_A \hat{A}_I\right)^{\otimes_{L^A \otimes_A \hat{A}_I} n}) \otimes_{L^A \otimes_A \hat{A}_I} \left(M \otimes_A \hat{A}_I\right) \\
&\cong (L^{\hat{A}_I} B)^{\otimes_{L^{\hat{A}_I}} n}) \otimes_{L^{\hat{A}_I}} \hat{M}_I,
\end{aligned}$$

which is the module of n-cochains $C^n_{(L^{\hat{A}_I}, L^{\hat{A}_I} B)}(\hat{M}_I)$. These isomorphisms are natural, commuting with the cobar complex differentials, hence giving us isomorphism 2.4.11. Meanwhile, isomorphism 2.4.10 follows from $\hat{A}_I$ being a flat A-module (see Proposition 10.14 in [1]), hence tensoring with $\hat{A}_I$ commutes with taking cohomology of the cobar complexes. □

3. The classifying rings L^A and $L^A B$ modulo torsion, for characteristic zero integral domains A.

In this section I compute L^A, for Dedekind domains A of characteristic zero, modulo torsion; that is, let $Q_A : \mathrm{Mod}(A) \to \mathrm{Mod}(A)$ be the functor that "quotients out" the maximal torsion submodule. The main result is Corollary 3.2.6, which gives a description of $Q_A(L^A)$ as well as the Hopf algebroid $(Q_A(L^A), Q_A(L^A B))$.

Remark 3.0.14. In [20], I compute L^A for rings of integers A in finite extensions of $\mathbb{Q}$. This does not render the results of the present section, computing $Q_A(L^A)$ for Dedekind domains A of characteristic zero, redundant: there are many Dedekind domains other than the usual examples, the rings of integers in number fields and function fields. See [5] for some exotic examples.

Remark 3.0.15. A computation of $Q_A(L^A)$ is a stronger result than a computation of $K(A) \otimes_A L^A$, that is, L^A base-changed up to the fraction field $K(A)$ of A, since the functor Q_A is less destructive than base-change to the fraction field. For example, let $A = \mathbb{Z}[x]$, let I be the ideal $(2, x)$ regarded as an A-module, and let $f : I \to A$ be the obvious inclusion. Then $K(A) \otimes_A f$ is an isomorphism, but $Q_A(f)$ is not.

More generally, the localization map $M \to K(A) \otimes_A M$ factors through the canonical map $M \to Q_A(M)$, so every map of A-modules which is inverted by Q_A is also inverted by base-change to the fraction field, but as the above example shows, the converse is not generally true.

3.1. **Killing torsion in commutative rings.** This subsection collects some basic properties of the operation of killing torsion in commutative rings. These facts are unsurprising and easy to prove and probably well-known, but I do not know of anywhere where they appear in print, so I collect them here. I make no claim that these little results are of any interest in their own right, but I use them all in the next subsection, in order to compute L^A modulo A-torsion in Theorem 3.2.4.

Definition-Proposition 3.1.1. *When A is a commutative ring and I an ideal of A, I will say that an element m of a left A-module M is* I-torsion *if there exists a nonzero element $i \in I$ such that $im = 0$. I will say that M itself is* I-torsion *if every element in M is I-torsion. I will say that a morphism $f : M \to N$ of A-modules* has I-torsion kernel *if the kernel* $\ker f$ *is I-torsion.*

When M is an A-module, I will write $T_I(M)$ for the sub-A-module of M consisting of all I-torsion elements of M. I will write $Q_I(M)$ for the quotient A-module $Q_I(M) = R/T_I(R)$. The functor Q_I preserves injections and also preserves surjections, but is not necessarily half-exact (and consequently, not necessarily exact).

If R is a commutative A-algebra, then $T_I(R)$ is an ideal in R, and hence $Q_I(R)$ is a quotient commutative A-algebra of R.

The special case $A = I$ arises sometimes (e.g. in Proposition 3.2.4); when $A = I$, we refer to I-torsion elements as simply torsion elements, *I-torsion modules as* torsion modules, *and so on.*

Proof. Suppose that $f : M \to N$ is a surjective A-module homomorphism, and suppose that $n \in Q_I(N)$. Choose a lift $\overline{n} \in N$ of n to N, and an element $m \in M$ such that $f(m) = \overline{n}$. Then $Q_I(f)(m) = n$, so $Q_I(f)$ is surjective.

Suppose instead that $f : M \to N$ is an injective A-module homomorphism, and suppose that $m \in \ker Q_I(f)$. Choose a lift $\overline{m} \in M$ of m to M. Since $Q_I(f)(m) = 0$, there exists some nonzero $i \in I$ such that $i \cdot f(\overline{m}) = 0$. So $f(i\overline{m}) = 0$, so $i\overline{m} = 0$, so $\overline{m} \in T_I(M)$ and $m = 0$. So $Q_I(f)$ is injective.

The classical example of Q_I failing to be half-exact is as follows: let $A = \mathbb{Z} = I$. Then

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

is exact, but applying Q_I yields $\mathbb{Z} \to \mathbb{Q} \to 0$, which is not exact. □

Lemma 3.1.2. *Let A be a commutative ring and let I be an ideal of A which has no internal zero divisors, i.e., if $ij = 0$ for some $i, j \in I$, then either $i = 0$ or $j = 0$. (This is weaker than asking that I contain no elements of A which are zero divisors in A.) Suppose that L, M, N are A-modules and suppose that $f : L \to M$ and $g : M \to N$ are A-module morphisms. If f and g each have I-torsion kernel, then their composite $g \circ f$ also has I-torsion kernel.*

Proof. We have the exact sequence

$$0 \to \ker f \xrightarrow{\iota} \ker g \circ f \xrightarrow{h} \ker g$$

hence the short exact sequence

$$0 \to \ker f \xrightarrow{\iota} \ker g \circ f \xrightarrow{h} \mathrm{im}\ h \to 0.$$

Now $\ker f$ is I-torsion by assumption, and $\mathrm{im}\ h$ is a sub-A-module of the I-torsion A-module $\ker g$, hence $\mathrm{im}\ h$ is I-torsion. For any element $x \in \ker g \circ f$, the element $h(x)$ is I-torsion (and possibly zero), so we can choose some nonzero $i \in I$ such that $h(ix) = 0$. So $ix \in \mathrm{im}\ \iota$, so choose $\tilde{x}$ such that $\iota\tilde{x} = ix$, and choose nonzero $j \in I$ such that $j\tilde{x} = 0$. Now

$ijx = 0$, and $ij \neq 0$ since I was assumed to have no internal zero divisors. So x is I-torsion, so $\ker g \circ f$ is I-torsion. □

Lemma 3.1.3. *Let A be a commutative ring and let I be an ideal of A which has no internal zero divisors, i.e., if $ij = 0$ for some $i, j \in I$, then either $i = 0$ or $j = 0$. Suppose that L, M, N are A-modules and suppose that $f : L \to M$ and $g : M \to N$ are A-module morphisms. Suppose that $g \circ f$ has I-torsion kernel. Then f has I-torsion kernel.*

Furthermore, if f is additionally assumed to be surjective, then g also has I-torsion kernel, and the A-module map $Q_I(f) : Q_I(L) \to Q_I(M)$ induced by f is an isomorphism.

Proof. It is elementary to show that the natural map $\ker f \to \ker g \circ f$ is a monomorphism, and that any submodule of an I-torsion module is itself I-torsion. So f has I-torsion kernel.

Now assume that f is also surjective. Then we have the short exact sequence

$$0 \to \ker f \to \ker g \circ f \to \ker g \to 0,$$

so $\ker g$ is a quotient of an I-torsion module. Again, it is elementary to show that a quotient of an I-torsion module is I-torsion, so g has I-torsion kernel.

Finally, we have the commutative square

(3.1.1)
$$\begin{tikzcd} L \arrow[r, "f"] \arrow[d, "\eta(L)"] & M \arrow[d, "\eta(M)"] \\ Q_I(L) \arrow[r, "Q_I(f)"] & Q_I(M) \end{tikzcd}$$

where the vertical maps $\eta(L)$ and $\eta(M)$ are the natural quotient maps. Since f is assumed surjective and since the right-hand map in diagram 3.1.1 is surjective, the bottom horizontal map $Q_I(f)$ is also surjective. Now $\eta(M) \circ f$ has I-torsion kernel by Lemma 3.1.2, hence both $Q_I(f) \circ \eta(L)$ and $\eta(L)$ have I-torsion kernel. Hence, using the parts of this very lemma that we have already proven, since $\eta(L)$ is surjective, we then have that $Q_I(f)$ has I-torsion kernel. But the only I-torsion submodule of $Q_I(L)$ is trivial, so $Q_I(f)$ is injective. So $Q_I(f)$ is an isomorphism as desired. □

Lemma 3.1.4. *If A is an integral domain of characteristic zero and M is an A-module, then the natural map*

$$f : \mathbb{Q} \otimes_{\mathbb{Z}} (Q_A(M)) \to Q_{\mathbb{Q}\otimes_{\mathbb{Z}}A}(\mathbb{Q} \otimes_{\mathbb{Z}} M)$$

is an isomorphism of $\mathbb{Q} \otimes_{\mathbb{Z}} A$-modules.

Proof. The map f is adjoint to the natural map of A-modules

$$f^\flat : Q_A(M) \to Q_{\mathbb{Q}\otimes_{\mathbb{Z}}A}(\mathbb{Q} \otimes_{\mathbb{Z}} M)$$

and an element $m \in Q_A(M)$ is in the kernel of $f^\flat$ if and only if m lifts to an element $\overline{m} \in M$ whose image in $\mathbb{Q} \otimes_{\mathbb{Z}} M$ is $\mathbb{Q} \otimes_{\mathbb{Z}} A$-torsion. Choose an element $a/p^n \in \mathbb{Q} \otimes_{\mathbb{Z}} A$ such that $(a/p^n)\overline{m} = 0 \in \mathbb{Q} \otimes_{\mathbb{Z}} M$ and such that $a \in A$. Then $0 = p^n(a/p^n\overline{m}) = a\overline{m}$ and hence $\overline{m} \in M$ is A-torsion, hence $m = 0 \in Q_A(M)$. So $f^\flat$ is injective, and since rationalization is an exact functor, $\mathbb{Q} \otimes_{\mathbb{Z}} f^\flat$ is injective, i.e. (since the codomain of $f^\flat$ is already rational), f is injective.

If $x \in Q_{\mathbb{Q}\otimes_{\mathbb{Z}}A}(\mathbb{Q} \otimes_{\mathbb{Z}} M)$, choose a lift of x to $\overline{x} \in \mathbb{Q} \otimes_{\mathbb{Z}} M$. Then $\overline{x} = y/p^n$ for some $y \in M$. Let $\tilde{y}$ be the image of y in $Q_A(M)$, and then $x = f(\tilde{y}/p^n)$. So f is surjective. □

For the proof of Lemma 3.2.2 it is necessary to prove some properties of the symmetric algebra functor Sym_A, that is, the left adjoint of the forgetful functor from commutative graded A-algebras to graded A-modules. Here A is any commutative ring. This is an opportune time to introduce both the symmetric algebras and the Rees algebras, both of which are classical constructions, but for which we will need graded versions as well, which are slightly less classical:

Definition 3.1.5. *Let A be a commutative ring, I an ideal of A.*

- *By the* Rees algebra of *I, written* $\mathrm{Rees}_A(I)$, *I mean the commutative A-algebra* $\oplus_{n\geq 0} I^n\{t^n\} \subseteq A[t]$.
- *Let j be an integer. By the* j-suspended Rees algebra of *I, written* $\mathrm{Rees}^j_A(I)$, *I mean the commutative graded A-algebra whose underlying commutative A-algebra is* $\mathrm{Rees}_A(I)$, *and which is equipped with the grading in which the summand $I^n\{t^n\}$ is in grading degree jn.*

Now, more generally, let A be a commutative ring and let M be an A-module.

- *By the* symmetric algebra of *M, written* $\mathrm{Sym}_A(M)$, *I mean the commutative A-algebra $\oplus_{n\geq 0}(M^{\otimes_A n})_{\Sigma_n}$, where $(M^{\otimes_A n})_{\Sigma_n}$ is the orbit module under the action of the symmetric group Σ_n on $M^{\otimes_A n}$ given by permuting the tensor factors.*
- *Let j be an integer. By the* j-suspended symmetric algebra of *M, written* $\mathrm{Sym}^j_A(M)$, *I mean the commutative graded A-algebra whose underlying commutative A-algebra is* $\mathrm{Sym}_A(M)$, *and which is equipped with the grading in which the summand $(M^{\otimes_A n})_{\Sigma_n}$ is in grading degree jn.*

Remark 3.1.6. Although Sym_A preserves epimorphisms, it typically does not preserve monomorphisms; see section 6.2 of chapter III of [2].

It is worth explaining a bit of detail about the failure of Sym_A to preserve monomorphisms, in particular, how $\mathrm{Sym}_A(M)$ may fail to be torsion-free even if M is torsion-free. Suppose that A is an integral domain, and suppose that M, N are torsion-free A-modules, i.e., if $am = 0$ with $a \in A$ and $m \in M$ then $a = 0$ or $m = 0$, and similarly for N. Then it is an easy exercise to show that $M \otimes_A N$ is torsion-free if A is a hereditary domain, but $M \otimes_A N$ is not necessarily torsion-free if A has global dimension greater than one! An instructive example is to let $A = \mathbb{Z}[x]$, to let M be any maximal ideal in A, and to let $N = M$.

Consequently, even if M is torsion-free (for example, M can be an ideal in the integral domain A), $M^{\otimes_A n}$ is not necessarily torsion-free, and $(M^{\otimes_A n})_{\Sigma_n}$ can also fail to be torsion-free. So even if the fundamental functional $\sigma_n : L^A_{n-1}/D^A_{n-1} \to A$ (from Definition-Proposition 3.2.1) is injective, $\mathrm{Sym}_A\left(L^A_{n-1}/D^A_{n-1}\right)$ may fail to be torsion-free even though $\mathrm{Sym}_A(A)$ is a free A-module and hence torsion-free, and consequently the map $\mathrm{Sym}_A\left(L^A_{n-1}/D^A_{n-1}\right) \to \mathrm{Sym}_A(A)$ is not injective even though $L^A_{n-1}/D^A_{n-1} \to A$ is.

The situation is even a little worse still: there are circumstances when we can conclude that $\mathrm{Sym}_A(L^A_{n-1}/D^A_{n-1})$ is torsion-free, for example, when A is a Noetherian integral domain and L^A_{n-1}/D^A_{n-1} is isomorphic to an ideal in A (this happens frequently: see the material on the "fundamental functional" in [20] for some conditions that guarantee that L^A_{n-1}/D^A_{n-1} is isomorphic to an ideal in A), and that ideal is generated by a d-sequence, then $\mathrm{Sym}_A(L^A_{n-1}/D^A_{n-1})$ is isomorphic to $\mathrm{Rees}_A(L^A_{n-1}/D^A_{n-1})$ and hence torsion-free by the main result of [9]. This situation frequently occurs in practice, even when A is not hereditary. It still does not lead to a complete computation of L^A, however, since even when we know that $\mathrm{Sym}_A(L^A_{n-1}/D^A_{n-1})$ is torsion-free, it may well be the case that $\mathrm{Sym}_A(\coprod_{n\geq 2} L^A_{n-1}/D^A_{n-1}) \cong \bigotimes_A^{n\geq 2} \mathrm{Sym}_A(L^A_{n-1}/D^A_{n-1})$ has torsion, if A has global dimension ≥ 2! This is responsible for the classifying rings L^A of formal A-modules sometimes having torsion even when all of the modules of indecomposables L^A_{n-1}/D^A_{n-1} are torsion-free. (If it were not for these phenomena, this paper could have been much shorter.)

However, Lemma 3.1.7 and Proposition 3.1.9 each describe circumstances under which Sym_A sends maps with torsion kernel to maps with torsion kernel:

Lemma 3.1.7. *Let A be a commutative ring, and let I be an ideal of A. Suppose that I has no internal zero divisors, i.e., if $i, j \in I$ and $ij = 0$, then either $i = 0$ or $j = 0$. Regard I as an A-module. Then the natural map*

$$\mathrm{Sym}_A(I) \to \mathrm{Sym}_A(A), \tag{3.1.2}$$

induced by the A-module inclusion $I \subseteq A$, has I-torsion kernel.

Proof. First, suppose that M is an A-module, and let s_M denote the A-module homomorphism $s_M : I \otimes_A M \to IM$ given by $s_m(i \otimes m) = im$. Clearly s_M is surjective. If $x = \sum_{j=1}^n i_j \otimes m_j \in \ker s_M$ and all $i_j \in I$ are nonzero, then $\sum_{j=1}^n i_j m_j = 0$, and consequently

$$\begin{aligned} \left(\prod_{j=1}^n i_j\right) x &= \sum_{j=1}^n \left(\prod_{j=1}^n i_j\right) i_j \otimes m_j \\ &= \left(\prod_{j=1}^n i_j\right) \otimes \left(\sum_{j=1}^n i_j m_j\right) \\ &= 0, \end{aligned}$$

and $\prod_{j=1}^n i_j = 0$ by the assumption that I has no internal zero divisors. So s_m has I-torsion kernel.

Now an obvious induction then shows that the kernel of the map $\tilde{c}_n : I^{\otimes_A n} \to I^n$, which sends $i_1 \otimes \cdots \otimes i_n$ to the product $i_1 \cdot \cdots \cdot i_n$, is I-torsion. So we have the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \ker \tilde{c}_n \arrow[r] \arrow[d, "q'"] & I^{\otimes_A n} \arrow[r, "\tilde{c}_n"] \arrow[d, "q"] & I^n \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \arrow[d] \\ 0 \arrow[r] & \ker c_n \arrow[r] & (I^{\otimes_A n})_{\Sigma_n} \arrow[r, "c", "n"'] & I^n \arrow[r] & 0 \end{tikzcd}$$

in which the quotient map q is surjective, hence (by e.g. the "four lemma" in homological algebra) the map q' is also surjective. Now $\ker \tilde{c}_n$ is an I-torsion A-module, and now we see that $\ker c_n$ is a quotient of $\ker \tilde{c}_n$, so $\ker c_n$ is also an I-torsion A-module. Hence c_n is surjective with I-torsion kernel. On taking direct sums, we have that the kernel of the map $\mathrm{Sym}_A(I) \cong \coprod_{n \geq 0} (I^{\otimes_A n})_{\Sigma_n} \to \coprod_{n \geq 0} I^n$ is $\coprod_{n \geq 0} \ker c$ (this uses the fact that the category of modules over a ring satisfies Grothendieck's axiom AB4, from [6], so infinite coproducts commute with kernels!). Finally, it is an easy exercise to show that the coproduct of a family, even an infinite family, of I-torsion A-modules is still I-torsion (although the same is not necessarily true for products). So the kernel of the map $f : \mathrm{Sym}_A(I) \to \coprod_{n \geq 0} I^n$, which sends $i_1 \otimes \ldots i_n$ to the product $i_1 \cdot \cdots \cdot i_n$, is I-torsion. Composing f with the coproduct of the inclusions $I^n \hookrightarrow A$ yields exactly the map

$$\mathrm{Sym}_A(I) \to \coprod_{n \geq 0} A \cong \coprod_{n \geq 0} \left(A^{\otimes_A n}\right)_{\Sigma_n} \cong \mathrm{Sym}_A(A).$$

induced by applying the functor Sym_A to the inclusion $I \hookrightarrow A$. Consequently the kernel of this induced map $\mathrm{Sym}_A(I) \to \mathrm{Sym}_A(A)$ is an I-torsion A-module. ◻

Lemma 3.1.8. *Let A be a commutative ring, let M be a nonzero A-module, and let $f : M \to A$ be a homomorphism of A-modules. Let I be an ideal of A which contains the image of f. Suppose that I has no internal zero divisors, i.e., if $i, j \in I$ and $ij = 0$, then* *either $i = 0$ or $j = 0$. Suppose also that f has I-torsion kernel. Finally, let q be a positive integer. Then the natural map*

$$f^{\otimes_A q} : M^{\otimes_A q} \rightarrow A^{\otimes_A q} \tag{3.1.3}$$

also has I-torsion kernel.

Proof. For any A-module N, the kernel of the map $f \circ N : M \otimes_A N \rightarrow A \otimes_A N$ is an I-torsion A-module. The proof of this claim is as follows: first, write K for the kernel and J for the image of the map f, so that

$$0 \rightarrow K \rightarrow M \xrightarrow{\tilde{f}} J \rightarrow 0$$

is short exact, where $\tilde{f}(m) = f(m)$. Since K is assumed to be I-torsion, we have that $\tilde{f} \otimes_A A[(I\backslash\{0\})^{\pm 1}]$ is an isomorphism, and since localization commutes with Tor (see e.g. Proposition 3.2.9 in [26]), $\tilde{f}$ induces an isomorphism

$$\mathrm{Tor}_r^{A[(I\backslash\{0\})^{\pm 1}]}(M[(I\backslash\{0\})^{\pm 1}], N[(I\backslash\{0\})^{\pm 1}]) \xrightarrow{\cong} \mathrm{Tor}_r^{A[(I\backslash\{0\})^{\pm 1}]}(J[(I\backslash\{0\})^{\pm 1}], N[(I\backslash\{0\})^{\pm 1}])$$

for all $r \geq 0$, and so

$$\mathrm{Tor}_r^A(K, N)[(I\backslash\{0\})^{\pm 1}] \cong \mathrm{Tor}_r^{A[(I\backslash\{0\})^{\pm 1}]}(K[(I\backslash\{0\})^{\pm 1}], N[(I\backslash\{0\})^{\pm 1}]) \cong 0$$

for all $r \geq 0$. Hence $\mathrm{Tor}_r^A(K, N)$ is an I-torsion A-module for all $r \geq 0$. In particular, $\mathrm{Tor}_1^A(K, N)$ is an I-torsion A-module, and the kernel of the map $\tilde{f} \otimes_A N$ is a quotient of $\mathrm{Tor}_1^A(K, N)$. So $\tilde{f} \otimes_A N$ has I-torsion kernel.

Similarly, we have the short exact sequence

$$0 \rightarrow J \xrightarrow{\iota} A \rightarrow A/M \rightarrow 0$$

where ι is the standard inclusion of J into A. Since J is contained in I by assumption, $\iota \otimes_A A[(I\backslash\{0\})^{\pm 1}]$ is also an isomorphism, so again, since localization commutes with Tor, we have that $\mathrm{Tor}_r^A(A/M, N)[(I\backslash\{0\})^{\pm 1}] \cong \mathrm{Tor}_r^{A[(I\backslash\{0\})^{\pm 1}]}(A/M[(I\backslash\{0\})^{\pm 1}], N[(I\backslash\{0\})^{\pm 1}])$ is trivial for all $r \geq 0$. Hence $\mathrm{Tor}_1^A(A/M, N)$ is I-torsion, hence its quotient, the kernel of $\iota \otimes_A N$, is also I-torsion.

Now $f = \iota \circ \tilde{f}$, and we have just shown that both ι and $\tilde{f}$ have I-torsion kernel, so Lemma 3.1.2 tells us that f also has I-torsion kernel, as claimed.

Now back to the proof. The map $f^{\otimes_A q}$ can be written as a composite

$$\begin{aligned} M \otimes M \otimes_A M \otimes_A \dots M &\xrightarrow{f \otimes \mathrm{id} \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}} A \otimes_A M \otimes_A M \otimes_A \cdots \otimes_A M \\ &\xrightarrow{\mathrm{id} \otimes f \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}} A \otimes_A A \otimes_A M \otimes_A \cdots \otimes_A M \\ &\xrightarrow{\mathrm{id} \otimes \mathrm{id} \otimes f \otimes \cdots \otimes \mathrm{id}} \dots \\ &\xrightarrow{\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \cdots \otimes f} A \otimes_A A \otimes_A A \otimes_A \cdots \otimes_A A \end{aligned}$$

and we have just shown that each of the maps in the composite has I-torsion kernel! We use Lemma 3.1.2 again, and we now have that $f^{\otimes_A q}$ has I-torsion kernel. □

Proposition 3.1.9. *Let A be an integral domain of characteristic zero, and let $f : M \rightarrow A$ be a homomorphism of A-modules. Suppose also that f has torsion kernel. Finally, let q be a positive integer. Then the natural map*

$$\mathrm{Sym}_A(f) : \mathrm{Sym}_A(M) \rightarrow \mathrm{Sym}_A(A) \tag{3.1.4}$$

also has torsion kernel.

Proof. By the $I = A$ case of Lemma 3.1.8, for each positive integer n, the kernel of the map $f^{\otimes_A n} : M^{\otimes_A n} \to A^{\otimes_A n}$ is torsion. Let V_n, W_n be the kernel and cokernel, respectively, of $f^{\otimes_A n}$, and let K be the fraction field of A. Then we have the short exact sequence of $A[\Sigma_n]$-modules

$$0 \to V_n \to M^{\otimes_A n} \to \text{im } f^{\otimes_A n} \to 0$$

and we know that

- localization is exact,
- localization commutes with taking Σ_n-orbits,
- $H_1(\Sigma_n; V)$ vanishes if $n!$ acts invertibly on V (this part is why we had to assume that A is characteristic zero and also why we are only trying to show that $\text{Sym}_A(f)$ has torsion kernel, not that $\text{Sym}_A(f)$ has I-torsion kernel for some proper ideal I of A; inverting all primes in $\mathbb{Z} \subseteq A$ is a cheap way to get the Σ-orbits functor to be exact. It may be possible to prove the same result inverting a smaller ideal, but only with significantly more work),
- and V_n is torsion,

hence we have isomorphisms

$$\begin{aligned} M^{\otimes_A n}_{\Sigma_n} \otimes_A K &\cong (M^{\otimes_A n} \otimes_A K)_{\Sigma_n} \\ &\cong (\text{im } f^{\otimes_A n}) \otimes_A K)_{\Sigma_n} \\ &\cong (\text{im } f^{\otimes_A n})_{\Sigma_n} \otimes_A K, \end{aligned}$$

hence the kernel of the map

$$(f^{\otimes_A n})_{\Sigma_n} : (M^{\otimes_A n})_{\Sigma_n} \to ((\text{im } f^{\otimes_A n})_{\Sigma_n}$$

is torsion.

Similarly we have the short exact sequence

$$0 \to \text{im } f^{\otimes_A n} \to A^{\otimes_A n} \to W_n \to 0$$

of $A[\Sigma_n]$-modules, and for the same reasons, $H_1(\Sigma_n; W_n \otimes_A K) \cong 0$. By standard base-change properties of Tor (see e.g. Corollary 3.2.10 of [26]), $H_1(\Sigma_n; W_n \otimes_A K) \cong H_1(\Sigma_n; W_n) \otimes_A K$, hence $H_1(\Sigma_n; W_n)$ is A-torsion. Since the kernel of the map $(\text{im } f^{\otimes_A n})_{\Sigma_n} \to (A^{\otimes_A n})_{\Sigma_n}$ is a quotient of the torsion A-module $H_1(\Sigma_n; W_n)$, the map $(\text{im } f^{\otimes_A n})_{\Sigma_n} \to (A^{\otimes_A n})_{\Sigma_n}$ has torsion kernel.

Now the map $(f^{\otimes_A n})_{\Sigma_n}$ is the composite of the two maps $(M^{\otimes_A n})_{\Sigma_n} \to (\text{im } f^{\otimes_A n})_{\Sigma_n}$ and $(\text{im } f^{\otimes_A n})_{\Sigma_n} \to (A^{\otimes_A n})_{\Sigma_n}$, each of which we have now shown to have torsion kernel, so by Lemma 3.1.2, $(f^{\otimes_A n})_{\Sigma_n}$ has torsion kernel.

Finally, the map $\text{Sym}_A(f) : \text{Sym}_A(M) \to \text{Sym}_A(A)$ is, up to isomorphism, the direct sum

$$\text{Sym}_A(M) \xrightarrow{\cong} \coprod_{n \geq 0} (M^{\otimes_A n})_{\Sigma_n} \xrightarrow{\coprod_{n\geq 0}(f^{\otimes_A n})_{\Sigma_n}} \coprod_{n \geq 0} (A^{\otimes_A n})_{\Sigma_n} \xrightarrow{\cong} \text{Sym}_A(A).$$

Since coproducts commute with kernels in the category of modules over a ring, and since a coproduct of torsion modules is a torsion module, $\text{Sym}_A(f)$ has torsion kernel. □

3.2. **Computation of L^A modulo torsion.** For the statements of Definition-Proposition 3.2.1, recall that L^A_n / D^A_n is the the grading degree n summand of L^A modulo the A-submodule generated by all products xy of homogeneous elements $x, y \in L^A$ of grading degree $< n$.

In Definition-Proposition 3.2.1 I define the “fundamental functional.” This is a new definition.

Definition-Proposition 3.2.1. *Let A be a commutative ring and let n be a positive integer. Recall from Proposition 1.2.2 that L^A_{n-1}/D^A_{n-1} is described by Drinfeld's presentation: it is generated, as an A-module, by elements d and $\{c_a\}_{a\in A}$, subject to the relations 1.2.1, 1.2.2, and 1.2.3.*

Let M^A_{n-1} denote the A-module generated by elements d and $\{c_a\}_{a\in A}$, subject only to the relations 1.2.1. Let $q^A_{n-1} : M^A_{n-1} \to L^A_{n-1}/D^A_{n-1}$ denote the obvious A-module quotient map.

*By the n*th fundamental functional of A, *I mean the A-module homomorphism*

$$\sigma_n : L^A_{n-1}/D^A_{n-1} \to A$$

given by

$$\sigma_n(d) = \nu(n),$$
$$\sigma_n(c_a) = a - a^n,$$

where $\nu(n)$ is as in the proof of Proposition 2.3.3, that is, $\nu(n) = p$ if n is a power of the prime p, and $\nu(n) = 1$ if n is not a prime power.

If $n > 1$, then the kernel of the composite map $\sigma_n \circ q^A_{n-1} : M^A_{n-1} \to A$ is exactly the set of $\nu(n)$-torsion elements of M^A_{n-1}. Furthermore, the kernel of σ_n and the kernel of q^A_{n-1} are each annihilated by multiplication by $\nu(n)$. Furthermore, if n is not a prime power, then σ_n and q^A_{n-1} are both isomorphisms of A-modules.

Proof. By the definition of M^A_{n-1}, the top row in the commutative diagram of A-modules

$$\begin{tikzcd}
\coprod_{a\in A} A\{r_a\} \arrow[r, "\delta_0"] & A\{d\} \oplus \coprod_{a\in A} A\{c_a\} \arrow[r, "\delta_{-1}"] \arrow[dr, "\tilde{\sigma}_n"'] & M^A_{n-1} \arrow[r] \arrow[d, "q^A_{n-1}\circ\sigma_n"] & 0 \\
 & & A &
\end{tikzcd}$$

is exact, where $\tilde{\sigma}_n$ is the map given by $\tilde{\sigma}_n(d) = \nu(n)$ and $\tilde{\sigma}_n(c_a) = a - a^n$, where δ_0 is the map given by $\delta_0(r_a) = (a - a^n)d - \nu(n)c_a$, and where δ_{-1} sends d to d and sends c_a to c_a. Suppose that $x = \beta d + \sum_{a\in A} \alpha_a c_a$ is in the kernel of $\tilde{\sigma}_n$, where $\beta \in A$ and $\alpha_a \in A$ for all $a \in A$. Then:

$$\begin{aligned} 0 &= \tilde{\sigma}_n(x) \\ &= \beta\nu(n) + \sum_{a\in A} \alpha_a(a - a^n), \end{aligned}$$

so $\beta\nu(n) = -\sum_{a\in A} \alpha_a(a - a^n)$. Let $\gamma_a = -\alpha_a$ for all $a \in A$. Then:

$$\begin{aligned} \delta_0\left(\sum_{a\in A} \gamma_a r_a\right) &= \left(\sum_{a\in A} \gamma_a(a - a^n)\right) d - \sum_{a\in A} \gamma_a \nu(n) c_a \\ &= \beta\nu(n)d + \sum_{a\in A} \nu(n)c_a \\ &= \nu(n)x, \end{aligned}$$

so $\nu(n)x \in \text{im } \delta_0$. Hence every element in the kernel of $q^A_{n-1} \circ \sigma_n$ is killed by $\nu(n)$.

Conversely, if $x = \beta d + \sum_{a\in A} \alpha_a c_a$ has the property that $\nu(n)x \in \text{im } \delta_0$, then there exists some $\gamma_a \in A$ for each $a \in A$ such that

$$\begin{aligned}\delta_0\left(\sum_{a\in A} \gamma_a r_a\right) &= \left(\sum_{a\in A} \gamma_a (a - a^n)\right) d - \sum_{a\in A} \gamma_a \nu(n) c_a \\ &= \nu(n)\beta d + \sum_{a\in A} \nu(n)\alpha_a c_a,\end{aligned}$$

i.e., $\gamma_a = -\alpha_a$ and

$$\beta\nu(n) = \sum_{a\in A} \gamma_a(a - a^n) = \sum_{a\in A} -\alpha_a(a - a^n),$$

and consequently

$$\tilde{\sigma}_n(x) = \beta\nu(n) + \sum_{a\in A} \alpha_a(a - a^n) = 0.$$

So every element in M^A_{n-1} killed by $\nu(n)$ is also in the kernel of $q^A_{n-1} \circ \sigma_n$.

By its construction, q^A_{n-1} is a surjection, and so we have a short exact sequence

$$0 \to \ker q^A_{n-1} \to \ker q^A_{n-1} \circ \sigma_n \to \ker \sigma_n \to 0.$$

We have just shown that every element in $\ker q^A_{n-1} \circ \sigma_n$ is killed by multiplication by $\nu(n)$, and now we see that $\ker \sigma_n$ is a quotient of $\ker q^A_{n-1} \circ \sigma_n$. Hence every element in $\ker \sigma_n$ is killed by multiplication by $\nu(n)$, as claimed. Similarly, $\ker q^A_{n-1}$ is a submodule of a module killed by multiplication by $\nu(n)$, so $\ker q^A_{n-1}$ is killed by multiplication by $\nu(n)$, as claimed.

Now suppose that n is not a prime power. We have just shown that the kernel of σ_n is killed by multiplication by $\nu(n)$, so if n is not a prime power then σ_n is injective. Furthermore, $\nu(n) = 1$ implies that $\sigma_n(d) = 1 \in A$, so σ_n is surjective. So σ_n is an isomorphism. One also checks easily that, when $\nu(n) = 1$, the relations 1.2.2 and 1.2.3 can be derived from the relation 1.2.1, so q^A_{n-1} is also an isomorphism. □

Lemma 3.2.2. *When R is a commutative ring and n is a positive integer, I will write R_n for the grading degree n summand of R_n, and $D(R)_n$ for the sub-R_0-module of R_n consisting of all elements of the form xy, where x, y are homogeneous elements of R of grading degree $< n$.*

Now let A be an integral domain of characteristic zero, let R, S be commutative graded A-algebras concentrated in nonnegative grading degrees, and let $f : R \to S$ be a graded A-algebra homomorphism. Suppose that S is a polynomial algebra over A, $S \cong A[x_1, x_2, \dots]$, with at most one x_i in each grading degree. Then, for each n, $S_n/D(S)_n$ is either trivial or a free A-module on one generator. Write $f_n : R_n/D(R)_n \to S_n/D(S)_n \cong A$ for the A-module map induced by f, and f_0 for the map $f_0 : R_0 \to S_0$ induced by f. Suppose that, for each n, the A-module projection $R_n \to R_n/D(R)_n$ splits (e.g. we could assume that $R_n/D(R)_n$ is projective for all n). For each n, write I_n for the kernel of f_n. Write I for the sum of the ideals $I = \sum_{n\geq 0} I_n \subseteq A$. Suppose that I has no internal zero divisors, i.e., if $ij = 0$ for some $i, j \in I$, then either $i = 0$ or $j = 0$. Then $Q_I(R)$ is isomorphic, as a commutative graded A-algebra, to Q_I applied to the symmetric algebra $\mathrm{Sym}_A\left(R_0 \oplus \coprod_{n\geq 1} R_n/D(R)_n\right)$. Furthermore, the kernel of f is I-torsion, i.e., for each $x \in \ker f$ there exists some nonzero $i \in I$ such that $ix = 0$.

Proof. Since S is polynomial with at most one generator in each grading degree, the A-module $S_n/D(S)_n$ is either trivial or isomorphic to A. Hence, for each n, either $R_n/D(R)_n$

is trivial or the map $f_n : R_n/D(R)_n$ has I-torsion kernel. Using Lemma 3.1.9, the upper horizontal map in the commutative diagram of commutative graded A-algebras

(3.2.1)
$$\begin{tikzcd} \mathrm{Sym}_A\left(R_0 \oplus \coprod_{n\geq 1} R_n/D(R)_n\right) \arrow[r] \arrow[d] & \mathrm{Sym}_A\left(S_0 \oplus \coprod_{n\geq 1} S_n/D(S)_n\right) \arrow[d] \\ R \arrow[r, "f"] & S \end{tikzcd}$$

has I-torsion kernel. The right-hand vertical map in diagram 3.2.1 is an isomorphism since S is polynomial, so the composite map $\mathrm{Sym}_A\left(R_0 \oplus \coprod_{n\geq 1} R_n/D(R)_n\right) \to S$ has I-torsion kernel, so the left-hand vertical map in diagram 3.2.1 has I-torsion kernel by Lemma 3.1.3. The left-hand vertical map in diagram 3.2.1 is also surjective, since every element in $D(R)_n$ is a product of elements of lower grading degree, So the bottom horizontal map, f itself, has I-torsion kernel, by Lemma 3.1.3. Furthermore, since the left-hand vertical map in diagram 3.2.1 is surjective and has I-torsion kernel, the graded A-algebra map

$$Q_I\left(\mathrm{Sym}_A\left(R_0 \oplus \coprod_{n\geq 1} R_n/D(R)_n\right)\right) \to Q_I(R)$$

it induces is an isomorphism of A-modules by Lemma 3.1.3, hence a graded A-algebra isomorphism, as desired. □

In Lemma 3.2.2, the assumption that the codomain algebra S is polynomial actually matters: otherwise there are easy counterexamples like $R = \mathbb{Z}[x] \to \mathbb{Z}[x]/x^2 = S$.

Lemma 3.2.3. *Let A be a commutative ring and let R be a commutative graded A-algebra concentrated in nonnegative degrees. Then, for each nonnegative integer n, the natural map of $\mathbb{Q} \otimes_{\mathbb{Z}} A$-modules*

$$f : \mathbb{Q} \otimes_{\mathbb{Z}} (R_n/D(R)_n) \to (\mathbb{Q} \otimes_{\mathbb{Z}} R)_n / D(\mathbb{Q} \otimes_{\mathbb{Z}} R)_n$$

is an isomorphism.

Proof. This proof closely resembles that of Lemma 3.1.4. The map f is adjoint to the natural map of A-modules

$$f^\flat : R_n/D(R)_n \to (\mathbb{Q} \otimes_{\mathbb{Z}} R)_n / D(\mathbb{Q} \otimes_{\mathbb{Z}} R)_n$$

and an element $r \in R_n/D(R)_n$ is in the kernel of $f^\flat$ if and only if r lifts to an element $\overline{r} \in R_n$ whose image $\tilde{r}$ in $\mathbb{Q} \otimes_{\mathbb{Z}} R$ satisfies an equality $\tilde{r} = xy$ where x, y are homogeneous elements of $\mathbb{Q} \otimes_{\mathbb{Z}} R$ of degree less than n. Then $x = x'/p^i$ and $y = y'/p^j$ for some integers i, j and some elements $x', y' \in R$. Hence $p^{i+j}\tilde{r} = x'y' \in D(R)_n$, and hence $p^{i+j}r = 0 \in R_n/D(R)_n$. Hence r is killed by the localization map $R_n/D(R)_n \to \mathbb{Q} \otimes_{\mathbb{Z}} R_n/D(R)_n$. Exactness of rationalization then implies that $\mathbb{Q} \otimes_{\mathbb{Z}} f^\flat$ is injective, i.e. (since the codomain of $f^\flat$ is already rational), f is injective.

Now suppose that $z \in (\mathbb{Q} \otimes_{\mathbb{Z}} R)_n / D(\mathbb{Q} \otimes_{\mathbb{Z}} R)_n$, and lift z to an element $r/p^i \in \mathbb{Q} \otimes_{\mathbb{Z}} R_n$ with $r \in R$ and i some integer. Then $f(r/p^i) = z$. So f is surjective. □

Theorem 3.2.4. *Let A be a Dedekind domain of characteristic zero. Then the graded A-algebra morphism*

$$Q_A(L^A) \to Q_A(L^{\mathbb{Q}\otimes_{\mathbb{Z}}A}),$$

obtained by applying Q_A (from Definition-Proposition 3.1.1) to the ring map $L^A \to L^{\mathbb{Q}\otimes_{\mathbb{Z}}A}$ classifying the underlying formal A-module of the universal formal $\mathbb{Q} \otimes_{\mathbb{Z}} A$-module, is *injective, and furthermore we have isomorphisms of graded A-algebras*

$$Q_A(L^A) \xrightarrow{\cong} Q_A\left(\mathrm{Sym}_A\left(\coprod_{n\geq 1} M_n^A\right)\right),$$

$$\xrightarrow{\cong} Q_A\left(\mathrm{Sym}_A\left(\coprod_{n\geq 1} L_n^A/D_n^A\right)\right) \tag{3.2.2}$$

$$\xrightarrow{\cong} Q_A\left(\mathrm{Sym}_A\left(\coprod_{n\geq 2} I_n^A\right)\right), \tag{3.2.3}$$

where M_n^A is defined as in Definition-Proposition 3.2.1, and where I_n^A is defined to be the ideal of A generated by $\nu(n)$ and all elements of A of the form $a^n - a$, where $\nu(n) = p$ if n is a power of a prime number p, and $\nu(n) = 1$ if n is not a prime power.

Furthermore, if i is an integer and we write $L^A_{\leq i}$ for the classifying ring of formal A-module i-buds, then we have isomorphisms of graded A-algebras

$$Q_A(L^A_{\leq i}) \xrightarrow{\cong} Q_A\left(\mathrm{Sym}_A\left(\coprod_{1\leq n\leq i} M_n^A\right)\right)$$

$$\xrightarrow{\cong} Q_A\left(\mathrm{Sym}_A\left(\coprod_{1\leq n\leq i} L_n^A/D_n^A\right)\right)$$

$$\xrightarrow{\cong} Q_A\left(\mathrm{Sym}_A\left(\coprod_{2\leq n\leq i+1} I_n^A\right)\right).$$

Proof. Let g denote the composite A-module morphism

$$L^A_{n-1}/D^A_{n-1} \xrightarrow{\sigma_n} A \hookrightarrow \mathbb{Q}\otimes_{\mathbb{Z}} A \xrightarrow{\cdot\frac{1}{\nu(n)}} \mathbb{Q}\otimes_{\mathbb{Z}} A,$$

where $A \hookrightarrow \mathbb{Q}\otimes_{\mathbb{Z}} A$ is the obvious localization map, and where $\cdot\frac{1}{\nu(n)}$ is the A-module isomorphism given by multiplication by $\frac{1}{\nu(n)}$. Then $g(d) = 1$ and $g(c_a) = \frac{a-a^n}{\nu(n)}$. This agrees with the map

$$L^A_{n-1}/D^A_{n-1} \to L^{\mathbb{Q}\otimes_{\mathbb{Z}}A}_{n-1}/D^{\mathbb{Q}\otimes_{\mathbb{Z}}A}_{n-1} \xrightarrow{h} \mathbb{Q}\otimes_{\mathbb{Z}} A,$$

induced by the localization map $A \to K(A)$, where h is the A-module isomorphism such that $h(d) = 1$ and $h(c_a) = \frac{a-a^n}{\nu(n)}$ for all $a \in A$ (see Proposition 1.2.3 for why this map is an isomorphism). The assumption that A is a characteristic zero integral domain implies that the underlying abelian group of A is torsion-free, and hence that the localization map $A \hookrightarrow \mathbb{Q}\otimes_{\mathbb{Z}} A$ is injective. Together with the fact (proven in Definition-Proposition 3.2.1) that σ_n has $\nu(n)$-torsion kernel, we have that g has torsion kernel. By Proposition 1.2.3, $L^{\mathbb{Q}\otimes_{\mathbb{Z}}A}$ is a graded polynomial algebra over $\mathbb{Q}\otimes_{\mathbb{Z}} A$, with one polynomial generator in each even positive grading degree. Consequently the assumptions of Lemma 3.2.2 are satisfied for the graded $\mathbb{Q}\otimes_{\mathbb{Z}} A$-algebra homomorphism

$$\mathbb{Q}\otimes_{\mathbb{Z}} L^A \to L^{\mathbb{Q}\otimes_{\mathbb{Z}}A}.$$

Lemma 3.2.2 then implies that the map

$$Q_I\left(\mathbb{Q}\otimes_{\mathbb{Z}} L^A\right) \to Q_I\left(L^{\mathbb{Q}\otimes_{\mathbb{Z}}A}\right)$$

is injective, and that $Q_I\left(\mathbb{Q}\otimes_{\mathbb{Z}} L^A\right)$ is isomorphic to $Q_I\left(\mathrm{Sym}_A\left(\coprod_{n\geq 0}(\mathbb{Q}\otimes_{\mathbb{Z}} L^A_n)/(\mathbb{Q}\otimes_{\mathbb{Z}} D^A_n)\right)\right)$, where $I = \mathbb{Q}\otimes_{\mathbb{Z}} A$. By Lemma 3.1.4 and Lemma 3.2.3, we now have that the natural map

$$\mathbb{Q}\otimes_{\mathbb{Z}} Q_A\left(\mathrm{Sym}_A\left(\coprod_{n\geq 0} L^A_n/D^A_n\right)\right)\to \mathbb{Q}\otimes_{\mathbb{Z}} Q_A(L^A) \tag{3.2.4}$$

is an isomorphism. Now the natural map $\mathrm{Sym}_A\left(\coprod_{n\geq 0} L^A_n/D^A_n\right)\to L^A$ is certainly surjective (this is just a rephrasing of the fact that every element of L^A is a product of elements in degree zero and indecomposable elements in positive degrees), hence since Q_A preserves surjections by Definition-Proposition 3.1.1,

$$Q_A\left(\mathrm{Sym}_A\left(\coprod_{n\geq 0} L^A_n/D^A_n\right)\right)\to Q_A(L^A) \tag{3.2.5}$$

is also surjective. It is elementary to show that a surjective homomorphism of torsion-free abelian groups which induces an isomorphism rationally is already an isomorphism. Since map 3.2.4 is the rationalization of the map 3.2.5, we now have that the map 3.2.5 is an isomorphism, as desired.

By Definition-Proposition 3.2.1, the maps $q^A_n : M^A_n \to L^A_n/D^A_n$ and $\sigma_{n+1} : L^A_n/D^A_n \to I^A_{n+1}$ are surjective with torsion kernel, and it is easy to see that a coproduct of surjective, torsion-kernel module morphisms is also surjective with torsion kernel, so by Proposition 3.1.9, q^A_n and σ_{n+1} each induce isomorphisms modulo torsion in Sym_A, i.e., the maps 3.2.2 and 3.2.3 are isomorphisms.

The claims for formal A-module buds are proven by the same line of argument as the claims for formal A-modules we have just proven. □

Remark 3.2.5. Because of the assumption that A is Dedekind, one could remove the need to apply Q (i.e., to reduce modulo torsion) in the statement of Theorem 3.2.4 if one simply knew that σ_n were injective for all n and all Dedekind domains A of characteristic zero. This is the general theme of [20], where I am able to show that σ_n is injective for all n and for all Dedekind domains A of characteristic zero *whose underlying abelian groups are finitely generated*, using homological methods. It seems reasonable to conjecture that σ_n remains injective even when A is not finitely generated, but I do not know a proof of this.

Corollary 3.2.6. *Let A be a Dedekind domain of characteristic zero. Let (L^A, L^AB) denote the graded Hopf algebroid classifying formal A-modules, as in Theorem 1.2.1. Let P denote the set of integers > 1 that are powers of prime numbers which are not invertible in A, and let R denote the set of integers > 1 not contained in P. Then we have an isomorphism of commutative graded A-algebras*

$$Q_A(L^A)\cong A[x_{n-1} : n\in R]\otimes_A \bigotimes_{A}^{n\in P}\mathrm{Rees}^{2n-2}_A\left(I^A_n\right),$$

where $Q_A(L^A)$ is the reduction of L^A modulo A-torsion, where each polynomial generator x_{n-1} is in grading degree $2(n-1)$, and where I^A_n is defined to be the ideal of A generated by $\nu(n)$ and all elements of A of the form $a^n - a$, and where $\nu(n) = p$ if n is a power of a prime number p, and $\nu(n) = 1$ if n is not a prime power. The classifying ring L^AB of strict isomorphisms of formal A-modules furthermore admits the isomorphism of commutative graded A-algebras

$$Q_A(L^AB)\cong (Q_A(L^A))[b_1, b_2, \dots],$$

where each polynomial generator b_i is in grading degree $2i$.

Proof. Immediate from Theorem 3.2.4 and Proposition 2.1.3. □

Corollary 3.2.7. *Let A be a Dedekind domain of characteristic zero. Then we have an isomorphism of commutative graded A-algebras*

$$Q_A(L^A) \cong A[x_{n-1} : n \in R] \otimes_A \bigotimes_A^{n \in P} \left(\bigotimes_A^{m \geq 1} A[y_{n^m}, z_{n^m}]/f_{n^m}(y_{n^m}, z_{n^m}) \right)$$

for some set of polynomials $\{f_{n^m}(y_{n^m}, z_{n^m}) : n \in P, m \geq 1\}$; here $Q_A(L^A)$ is the reduction of L^A modulo A-torsion, each polynomial generator x_{n-1} is in grading degree $2(n-1)$, and each polynomial generator y_{n^m}, z_{n^m} is in grading degree $2(n^m - 1)$.

Proof. Any ideals in any Dedekind domain A is generated by at most two elements i, j, and as an A-module the ideal has generators e_i and e_j and the single relation $ie_j = je_i$; so the Rees ring $\mathrm{Rees}_A(I_n^A)$ appearing in Corollary 3.2.6 is isomorphic to $A[e_i, e_j]/(ie_j - je_i)$. □

Recall that Drinfeld proved (as the corollary following Proposition 1.4 in [3]) the following two properties of formal A-modules when A is the ring of integers in a local nonarchimedean field:

All formal module buds extend: Every formal A-module n-bud extends to a formal A-module.

All formal modules lift: If R is a commutative A-algebra and I is an ideal of R, then every formal A-module over R/I is the modulo-I reduction of a formal A-module over R.

Now we are in a position to show that the same holds for formal A-module for A much more general than rings of integers in local fields:

Corollary 3.2.8. *Let A be a Dedekind domain of characteristic zero, and let R be a commutative A-algebra which is torsion-free as an A-module. Then the following statements are both true:*

All formal module buds extend: *Every formal A-module n-bud over R extends to a formal A-module.*

All formal modules lift: *If R is a commutative A-algebra and I is an ideal of R such that R/I is a torsion-free A-module, then every formal A-module over R/I is the modulo-I reduction of a formal A-module over R.*

Proof.
- If F is a formal A-module n-bud over R, then F extends to a formal A-module if and only if there exists a map filling in the dotted arrow to make the diagram

(3.2.6)
$$\begin{tikzcd} L^A_{\leq i} \arrow[r] \arrow[d] & R \\ L^A \arrow[ur, dashed] & \end{tikzcd}$$

commute, where the vertical map in diagram 3.2.6 is the map classifying the underlying i-bud of the universal formal A-module, and the horizontal map is the map classifying F. Since R is assumed to be torsion-free, every map from L^A to R factors through the quotient map $L^A \to Q_A(L^A)$, and similarly, every map from $L^A_{\leq i}$ to R factors through the quotient map $L^A_{\leq i} \to Q_A(L^A_{\leq i})$. By Theorem 3.2.4, the

vertical map in diagram 3.2.6 is, after applying Q_A, the map induced in Sym_A by the summand inclusion
$$\coprod_{1\leq n\leq i} M_n^A \hookrightarrow \coprod_{1\leq n} M_n^A,$$
hence a lift map filling in diagram 3.2.6 can always be chosen for R torsion-free by simply mapping all M_n^A to zero for $n > i$.

- Suppose $R \to R/I$ satisfies the conditions stated in the theorem, and suppose that F is a formal A-module over R/I. Since A is Dedekind, every ideal of A is projective as an A-module, so all I_n^A are projective A-modules. Hence every diagram of A-module morphisms of the form
$$\begin{tikzcd} \coprod_{1\leq n} I_n^A \arrow[d, dashed] \arrow[dr] & \\ R \arrow[r] & R/I \end{tikzcd}$$
has a map filling in the dotted arrow and making the diagram commute, by the universal property of projective modules (and the fact that it is preserved under coproduct). By the universal properties of Sym_A and Q_A, this is the same as saying that every diagram of A-torsion-free commutative A-algebras of the form
$$\begin{tikzcd} Q_A\left(\mathrm{Sym}_A\left(\coprod_{1\leq n} I_n^A\right)\right) \arrow[d, dashed] \arrow[dr] & \\ R \arrow[r] & R/I \end{tikzcd}$$
has a map filling in the dotted arrow and making the diagram commute, by the universal property of projective modules (and the fact that it is preserved under coproduct). Now by Theorem 3.2.4, $Q_A\left(\mathrm{Sym}_A\left(\coprod_{1\leq n} I_n^A\right)\right) \cong Q_A(L^A)$, and maps from L^A into A-torsion-free commutative A-algebras are in natural bijection with maps from $Q_A(L^A)$ into A-torsion-free commutative A-algebras. So every formal A-module over R/I lifts to R.

□

Remark 3.2.9. Given how well everything works in the setting of Corollary 3.2.8, one can ask if there any situation in which formal module n-buds are known to *not* extend to formal modules. Something along these lines does indeed happen: in unpublished work of J. Beardsley on "non-smooth formal groups," i.e., group structures on non-smooth formal affine schemes (rather than $\hat{\mathbb{A}}^1$ or $\hat{\mathbb{A}}^n$), some non-smooth formal group n-buds fail to extend to non-smooth formal group $(n+1)$-buds, and some deformation-theoretic description (e.g. an obstruction cocycle to extension) of this phenomenon is possible.

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