k a field

R k-algebra (commutative)

$\rightsquigarrow HH(R/k) = R \otimes^L_{R\otimes_k R} R$

$HH_*(R/k) = H_*(HH(R/k))$

$\rightsquigarrow$ if R smooth$/k$,

$HH_*(R/k) \simeq \Omega^*_{R/k}$

$\rightsquigarrow$ R a commutative k-alg, then $HH(R/k)$ is a simplicial comm k-alg w/ S^1-action

universal

$R \to HH(R/k)$

$\rightsquigarrow$ S^1-action on h'tpy groups recovers de Rham differentials

Can also allow k to be a more general ring

$$HH(R/k) = R \overset{L}{\otimes} R$$

$R \otimes^L_k R$ ← also derive k-linear tensor product

Try to make k "smaller"

Ex) if $k = \mathbb{F}_p$, consider $HH(R/\mathbb{F}_p)$ R $\mathbb{F}_p$-alg

Potentially more information $HH(R/\mathbb{Z})$

In fact, $HH(R/\mathbb{Z})$ determines $HH(R/\mathbb{F}_p)$

<u>Calculation:</u> $HH(\mathbb{F}_p/\mathbb{Z}_p)_* = \Gamma_{\mathbb{F}_p}(\sigma)$, $|\sigma| = 2$

divided power algebra

Idea of THH is to replace base by the sphere spectrum (initial object in E_∞-ring spectra).

Then divided powers go away!

<u>Def</u> R a comm ring,

$$THH(R) = R \underset{R \wedge R}{\wedge} R$$

$THH(R)$ is an $\mathcal{E}_\infty$-ring spectrum, equipped w/ an S^1-action & a map $R \to THH(R)$ $\Rightarrow$ initial for this structure. $\left(THH(R) = S^1 \otimes R\right)$

<u>Theorem</u> (Bökstedt):

$$THH(\mathbb{F}_p)_* = \mathbb{F}_p[\sigma], \qquad |\sigma| = 2.$$

$$THH(\mathbb{F}_p) = \mathbb{F}_p \underset{\mathbb{F}_p \wedge \mathbb{F}_p}{\wedge} \mathbb{F}_p$$

$\mathbb{F}_p \wedge \mathbb{F}_p \leftarrow$ dual Steenrod alg

$(\mathbb{F}_p \wedge \mathbb{F}_p)_* =$ dual Steenrod algebra

Can also be proved using the fact that $\mathbb{F}_p$ is a Thom spectrum. (get a formula for Thom spectra).

Rmk: If char. 0, (R is a $\mathbb{Q}$-alg),

then $THH(R) = HH(R/\mathbb{Q})$.

In char. p, (or mixed char.), you do get more information.

<u>Statement</u>: R an $\mathbb{F}_p$-alg

$THH(R)$ is a $THH(\mathbb{F}_p)$-module

& $THH(R) \wedge_{THH(\mathbb{F}_p)} \mathbb{F}_p \simeq HH(R/\mathbb{F}_p)$

Note that $THH(\mathbb{F}_p)_* = \mathbb{F}_p[\sigma]$, so this relative smash product is

$$\mathrm{cofib}(\sigma : \Sigma^2 THH(R) \to THH(R)) \simeq HH(R/\mathbb{F}_p).$$

$THH(R)$ is a "1-parameter deformation" of $HH(R/\mathbb{F}_p)$ (along parameter σ).

(Analogous to the fact that de Rham cohomology has a 1-parameter deformation given by crystalline cohomology, parameter p.)

THH has additional structure than the S^1-action.
Called a cyclotomic spectrum.
Can be expressed using language of G-equivariant spectra.

$\rightsquigarrow$ Can form $THH(R)^{C_{p^n}}$ for $n \geq 0$.

$\rightsquigarrow$ These are related to each other by certain maps F, V, R (Frobenius, Verschiebung, Restriction).

$$F: THH(R)^{C_{p^{n+1}}} \longrightarrow THH(R)^{C_{p^n}}$$
$$V: THH(R)^{C_{p^n}} \longrightarrow THH(R)^{C_{p^{n+1}}}$$
$$R: THH(R)^{C_{p^{n+1}}} \longrightarrow THH(R)^{C_{p^n}}$$

(come from the fact that one has a genuine equivariant spectrum).

relies on cyclotomic structure.

These maps look like the structure on the p-typical Witt vectors of a (comm ring).

Theorem (Hesselholt-Madsen): R a comm ring,
$$\pi_0\left(THH(R)^{C_{p^n}}\right) \simeq W_{n+1}(R) \quad \&$$

F, V, R correspond to Witt vector
Frobenius, Verschiebung, Restriction maps.

<u>Def</u> (Nikolaus–Scholze):
A <u>cyclotomic spectrum</u> is a spectrum X w/
$S^1 \curvearrowright X$ together w/ a map
("cyclotomic Frobenius")

$$\varphi_p : X \longrightarrow X^{tC_p}$$

$$\circlearrowleft_{S^1} \quad \cong \quad \circlearrowleft_{S^1/C_p}$$

equivariant map

Claim: $THH(R)$ is naturally a cyclotomic spectrum
($HH(R/k)$ generally does not have this structure).

Lets define a more [redacted] invariant of rings
than $HH(R/k)^{hS^1}$ $\rightsquigarrow$ can form
<u>topological cyclic homology</u> $TC(R) = \mathrm{Hom}_{CycSp}(\mathbb{1}, THH(R))$.

$\rightsquigarrow$ TC is relevant for algebraic K-theory.

Why does THH(R) have this additional structure (as a cyclotomic spectrum)?

R a commutative ring, we need to produce a map

$$\varphi_p : \quad THH(R) \longrightarrow THH(R)^{tC_p}$$

(S^1-equivariant)

Recall that $THH(R) = S^1 \otimes R$ (in $\mathbb{E}_\infty$-ring spectra)

To give an S^1-equiv map out of $THH(R)$, suffices to give a map out of R.

Constructing cyclotomic Frobenius is equivalent to

$$R \longrightarrow THH(R)^{tC_p} \quad \text{(as a map of } \mathbb{E}_\infty\text{-rings).}$$

Here it is essential to work over the sphere spectrum!

Consider the following setup.

M abelian group

$$\varphi : M \longrightarrow M^{\otimes p} \qquad x \longmapsto x \otimes x \otimes \cdots \otimes x$$

This is not a homomorphism, but can be corrected as follows.

$C_p \curvearrowright M^{\otimes p}$ permuting the factors.

$\varphi(x+y) \neq \varphi(x)+\varphi(y)$ but the discrepancy is a C_p-norm.

Ex) $p=2$.

$$(x+y)\otimes(x+y) = x\otimes x + y\otimes y + \underbrace{x\otimes y + y\otimes x}_{\text{norms. (norm of } x\otimes y)}$$

Construction M abelian group

$$\varphi : M \longrightarrow \frac{(M^{\otimes p})^{C_p}}{\text{image of norms}\left((M^{\otimes p})_{C_p} \to (M^{\otimes p})^{C_p}\right)} = \hat{H}^0(C_p, M^{\otimes p}).$$

$$x \mapsto x\otimes x\otimes \cdots \otimes x$$

If M is a commutative ring,

$\varphi : M \to \dfrac{(M^{\otimes p})^{C_p}}{\text{norms}}$ is a ring map

$$\downarrow$$

$$\frac{(M)^{C_p}}{\text{norms}} = M/p \qquad\qquad M^{\otimes p} \to M$$

Frobenius.

These types of maps are how one defines the cyclotomic Frobenius.

In the setting of spectra, one has an analog of these diagonal maps.

<u>Construction</u> If X a spectrum, there is a natural map

$$X \longrightarrow \left(X^{\wedge p}\right)^{tC_p}$$

version of the previous map for spectra.

Why? RHS is an exact, lax symmetric functor of X. Gives a natural map from $\mathrm{id} \longrightarrow \left(\ ^{\wedge p}\right)^{tC_p}$.

(Explained in the paper by Nikolaus-Scholze).

This does not work in derived category of $\mathbb{F}_p$.

i.e. $V \in D(\mathbb{F}_p)$

$$V \dashrightarrow \left(V^{\otimes p}\right)^{tC_p}$$

can't produce Frobenius map (as $\mathbb{F}_p$-linear map).

Return to THH $\longrightarrow$

R a comm ring,

Want

$$\begin{tikzcd}
R \arrow[rr, "\text{cyclotomic Frobenius}"] \arrow[dr, "\text{general Frobenius}"'] & & \mathrm{THH}(R)^{tC_p} \\
& \left(R^{\otimes p}\right)^{tC_p} \arrow[ur, "\text{formal}"'] &
\end{tikzcd}$$

b/c $R^{\otimes p} \xrightarrow{C_p\text{-equiv}} \mathrm{THH}(R)$

$C_p \subseteq S^1$

relies on the fact that over sphere spectrum

Examples:

$\mathrm{THH}(\mathbb{F}_p)_* = \mathbb{F}_p[\sigma], \ |\sigma| = 2$

$\mathrm{THH}(\mathbb{F}_p)^{tC_p} = \mathbb{F}_p[x^{\pm 1}], \ |x| = -2.$

The Frobenius

$$\varphi : THH(\mathbb{F}_p) \to THH(\mathbb{F}_p)^{tC_p}$$ is

an equivalence on connective covers. ($\sigma \mapsto x^{-1}$).

More generally, for any $\mathbb{F}_p$-algebra R,

$$THH(R) \to THH(R)^{tC_p} \simeq HP(R/\mathbb{F}_p)$$

requires some work.

This is a bit surprising, b/c

- $THH(R)$ (slogan: 1-parameter def of $HH(R/\mathbb{F}_p)$)
- RHS is $HP(R/\mathbb{F}_p) \to$ de Rham cohomology $R/\mathbb{F}_p$

If R is smooth over $\mathbb{F}_p$, then φ is an isomorphism in large degrees ($\geq \dim$).

This leads to the fact that $TC(R)$ is in bounded degrees.

<u>Thm</u> (Hesselholt): R smooth alg over a perfect field k, then

$$THH(R)_* \simeq \Omega^*_{R/k} \otimes_k k[\sigma] \qquad |\sigma| = 2.$$

& cyclotomic Frobenius

$$THH(R) \longrightarrow THH(R)^{tC_p} = HP(R/k)$$ is an equivalence in degrees $\geq$ dim.

("Cartier-type operator").

More generally, can calculate

$$THH(R)^{C_{p^n}}_* \simeq W_{n+1}\Omega_R [\sigma_n],$$

$|\sigma_n| = 2.$

R/k $\qquad HH(R/k)_{hS^1}$ "cyclic homology"

$\qquad HH(R/k)^{hS^1}$ "negative cyclic homology"

$TC(R)^\wedge_p$ in degrees ≥ -1

$HH(R/k)^{hS^1}$ not bounded

$$TC(R) = \mathrm{Hom}_{CycSp}(\mathbb{1}, THH(R))$$

$$TC(R)^{\wedge}_p = eq\left(THH(R)^{hS^1} \overset{can}{\underset{\varphi}{\rightrightarrows}} THH(R)^{\wedge\, tS^1} \right)$$

$\mathcal{C}$ stable ∞-category $\rightsquigarrow$ $THH(\mathcal{C})$ cyclotomic spectrum.

($\mathcal{C} = D(R)$, R a comm ring)

$\rightarrow$ can define S^1-action, TC, ...

In comm case, THH has universal property $S^1 \otimes$.

THH: stable ∞-categories $\longrightarrow$ cyclotomic spectra.

$$TC(R) = \varprojlim_{R, F} THH(R)^{C_{p^n}}$$

$$= \left(\varprojlim_{R} THH(R)^{C_{p^n}} \right)^{F}$$