THH & the description of

$THH(\mathbb{F}_p)$ → due to Bökstedt (unpublished), also done by Breen purely algebraically →

Still it seems that $THH(\mathbb{F}_p)_*$ as a ring, one needs some topological input, e.g. Steenrod operations.

See Franjou-Lannes-Schwartz.

For a reference, Blumberg-Cohen-Schlichtkrull → compute $THH(\mathbb{F}_p)$ using $\mathbb{F}_p$ as a Thom spectrum

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See also notes by Krause–Nikolaus for a discussion. (on THH).

Topological cyclic homology appeared in Bökstedt–Hsiang–Madsen,

$$K(R) \longrightarrow TC(R)$$

Theorem of Dundas–Goodwillie–McCarthy, if R a ring, $I \subseteq R$ nilpotent, have a cartesian square

$$\begin{array}{ccc} K(R) & \longrightarrow & TC(R) \\ \downarrow & & \downarrow \\ K(R/I) & \longrightarrow & TC(R/I) \end{array}$$

Applications by Hesselholt, Madsen to compute K-theory of lots of rings.

→ See Madsen's survey "Algebraic K-theory & traces"

Cyclotomic spectra

$$TC(R) = \mathrm{Hom}_{CycSp}(\mathbb{1}, THH(R))$$

Initially one didn't have CycSp as a homotopy theory. (instead TC defined more explicitly).

Mentioned by Kaledin, 2010.

The homotopy theory CycSp was constructed by Blumberg-Mandell & formula above is correct.

Also described by Barwick-Glasman, Ayala-Mazel-Gee-Rozenblyum

Nikolaus-Scholze: all descriptions of cyclotomic spectra have redundancy in bounded-below case.

They give a description of cyclotomic spectra,

$$\begin{array}{ccc} {}^{S^1}\circlearrowright X & \xrightarrow{\varphi} & X^{tC_p} \circlearrowleft {}^{S^1/C_p} \\ & = & \end{array}$$

The description of $THH(R)$, R a comm ring $\leadsto$ from their paper.

Antieau-Nikolaus: Give a description of cyclotomic spectra in terms of the invariant TR $\longrightarrow$ "topological Cartier modules"

<u>Main theorem</u> (Bhatt-Morrow-Scholze).

R smooth algebra / perfect field k

Then there is a filtration on the $\boxed{TP(R) = THH(R)^{tS^1}}$ whose associated graded is

(2-periodic) crystalline cohomology of R.

Ex) $R = \mathbb{F}_p$.

$$THH(\mathbb{F}_p)_* = \mathbb{F}_p[\sigma], \quad |\sigma| = 2.$$

$$TP(\mathbb{F}_p)_* = \mathbb{Z}_p[x^{\pm 1}], \quad |x| = -2.$$

crystalline coh of $\mathbb{F}_p$ is $\mathbb{Z}_p$.

<u>Rmk</u> R any $\mathbb{F}_p$-algebra

$TP(R)$ is a module over $TP(\mathbb{F}_p)$

$$TP(R) \otimes^L_{\mathbb{Z}_p} \mathbb{F}_p \simeq HP(R/\mathbb{F}_p)$$

Constructing this filtration is somewhat subtle (Antieau-Nikolaus gave a simpler construction); BMS really want to input mixed char. rings (e.g. $\mathbb{Z}_p$).

How to construct filtration on $TP(R)$?

1) Derive all functors so can apply to large $\mathbb{F}_p$-alg. (derive differential forms, dR coh).

2) Construct the filtration directly for large $\mathbb{F}_p$-alg ("regular semiperfect") (Postnikov filtration).

TP is here in even degrees.

3) Define on smooth $\mathbb{F}_p$-algs by faithfully flat descent.

Derived functors ("nonabelian").

Ex) k field

$\mathrm{Sym}^i : \mathrm{Vect}_k \to \mathrm{Vect}_k.$

Admits a derived functor

$\mathbb{L}\mathrm{Sym}^i : D(k)_{\geq 0} \longrightarrow D(k)_{\geq 0}.$

Explicitly $P_\cdot \rightsquigarrow$ Dold-Kan into a simplicial k-vector space $P^!_\bullet$

$\rightsquigarrow$ Consider $\mathrm{Sym}^i P^!_\cdot$ as a new simplicial k-vector space $\rightsquigarrow$ apply DK to a chain cx. This is $\mathbb{L}\mathrm{Sym}^i$.

Ex) (Quillen cotangent complex).

k a base field.

Each comm k-alg R, $\rightsquigarrow$ $\Omega^1_{R/k}$

Kähler differentials

The "derived functor" is the cotangent complex $L_{R/k}$.

Construction: R a ring, choose a simplicial k-alg $P_\bullet$ w/ $P_\bullet \longrightarrow R$ q-iso.

Then $L_{R/k} \simeq |\Omega^1_{P_\cdot/k}|$

geometric realization

More generally R can be a SCR.

Key property: $H_0(L_{R/k}) = \Omega^1_{R/k}$.

Can be higher homology $\rightsquigarrow$

does not happen if R/k <u>smooth</u>.

Consider $SCR_k \rightsquigarrow$ homotopy theory of simplicial comm rings

$Poly_k^{f.g.} \subseteq SCR_k$

<u>Fact:</u> If $\mathcal{C}$ is any ∞-category w/ sifted colimits,

$$\mathrm{Fun}(\mathrm{Poly}_k^{f.g.}, \mathcal{C}) \simeq \mathrm{Fun}'(\mathrm{SCR}_k, \mathcal{C})$$

commute w/ sifted colimits.

Ex) $\mathcal{C} = D(k)$.

F: $\Omega^1_{/k}$ on polynomial rings

(Analog of this for derived symmetric, exterior $\rightsquigarrow$ universal property for derived category).

In general, it's hard to make this explicit on rings (which are not polynomial).

Ex) Quillen cotangent cx

$$L_{R/k} \simeq \Omega^1_{R/k}$$

if R smooth.

Another example: <u>derived de Rham cohomology</u>.

Recall (k a base field)

R/k alg (smooth),

Consider $(\Omega^{\bullet}_{R/k}, d) \rightsquigarrow$ commutative dga over k.

smooth alg $/k \longrightarrow$ $\mathcal{E}_\infty$-alg over k

$R \longmapsto (\Omega^{\bullet}_{R/k}, d)$.

Let's try to derive this construction (Illusie):

Restrict polynomial rings, & then resolve (simplicial) rings by polynomial rings.

$dR_k : SCR_k \longrightarrow \mathcal{E}_\infty$-alg over k

"derived de Rham cohomology"

Ex) If R is a poly ring, it's usual de Rham complex $\longrightarrow$ in general based on some resolution..

Theorem (Bhatt): k perfect, char. p.

$$dR_{R/k} \simeq (\Omega^\bullet_{R/k}, d) \text{ if}$$

R smooth. (Cartier isomorphism conjugate filtration).

Not true in char. 0!

Also a theory of derived crystalline cohomology, agrees w/ ord. crystalline coh on smooth algs (k-alg).

Prop k base field, R/k alg.

Claim is that $HH(R/k)$ has a (convergent) descending filtration whose gr^i

$$\cong \left(\Lambda^i_R L_{R/k}\right)[i]$$

- consider $L_{R/k}$ is in $D(R)$
- Λ^i is (derived) ith exterior in R-modules.

Pf: Consider

$$HH(-/k): SCR_k \longrightarrow D(k).$$

Observe that this commutes w/ sifted colimits.

Everything is determined by polynomial rings.

In fact, $HH(-/k)$ is completely by value on polynomial rings.

If P is a polynomial ring,

HKR theorem: $HH_*(P/k) \simeq \Omega^*_{P/k}$.

(true for smooth algebras).

Take the Postnikov filtration

$$F^i\, HH(P/k) = \tau_{\geq i}\, HH(P/k)$$

$$gr^i = \left(\Lambda^i \Omega_{P/k}\right)[i].$$

Proved prop if P is polynomial, now extend formally (Kan extension) to get the statement in general. $\square$.

For this to be useful, need rings for which $L_{B/k}$ known.

<u>Prop:</u> k perfect, char. p.

R/k is a <u>perfect</u> ring

meaning Frobenius: $R \to R$ is an iso.

Then $L_{R/k} = 0$.

Ex) $\mathbb{F}_p[X^{1/p^\infty}]$ is a perfect ring.

Idea: in deg 0, $\Omega^1_{R/k}$,

$dx = 0$ $x \in R$. $x = y^p$

$dx = d(y^p) = p y^{p-1} dy = 0$ in char. p.

Cor: R/k perfect ring,

$$HH(R/k)_* \simeq R. \quad \text{(in deg 0)}.$$

Pf: Use HKR filtration from previous prop.; $L_{R/k} = 0$. $\square$.

Cor: $THH(R)_* = R[\sigma]$.

R perfect

Describe $THH(\mathbb{F}_p)_*^{hS^1} \simeq \dfrac{\mathbb{Z}_p[x, \sigma]}{x\sigma = p}$

$|\sigma| = 2$

$|x| = -2 \quad (x \in H^2(\mathbb{CP}^\infty))$

Nikolaus-Scholze $\longrightarrow$ $THH(\mathbb{F}_p)$ as a cyclotomic spectrum,

$$THH(\mathbb{F}_p) \simeq \tau_{\geq 0}\left(\mathbb{Z}_p^{tC_p}\right)$$

as E_∞-ring