Theorem (Bhatt–Morrow–Scholze):

R/k smooth, k perf. char. $p>0$.

Then $TP(R) := THH(R)^{tS^1}$ has a filtration whose $gr \simeq$ 2-periodic $R\Gamma_{crys}(\mathrm{Spec}(R))$

integral version of de Rham coh, $\simeq \widehat{\Omega^{*}_{\tilde{R}/W(k)}}$ whenever $\tilde{R}$ is a lift of R to $W(k)$

Supposed to be analogous to the motivic filtration on algebraic K-theory.

Ingredients in the proof of this theorem

Last time, talked about machinery of derived functors & derived de Rham cohomology, cotangent complex

$$\begin{array}{cc} \text{de Rham cohomology} & \text{cotangent complex} \\ \rightsquigarrow & \rightsquigarrow \\ dR_{/k} & L_{/k}. \end{array}$$

Prop R/k any alg, then there is a filtration on $HH(R/k)$ whose associated gr is $\left(\bigwedge^i_R L_{R/k}\right)[i]$

$\rightsquigarrow$ Pf: Postnikov filtration for R smooth, derived functors everywhere.

Ex) $HH(R/k)_* = R$ if R perfect $\mathbb{F}_p$-alg (k perfect field, char. p.)

Cor If R is a perfect $\mathbb{F}_p$-alg,

$$THH(R)_* = R[\sigma], \quad |\sigma| = 2.$$

(b/c $THH(R)/\sigma \simeq HH(R/\mathbb{F}_p) = R$)

Theorem (BMS):

The construction $R \longmapsto THH(R)$ satisfies faithfully flat descent as does $R \longmapsto TP(R)$

$TC^-(R) = THH(R)^{hS^1}$

...

More precisely, if $R \to S$, then

$$TP(R) \simeq \mathrm{Tot}\left(TP(S) \rightrightarrows TP(S \otimes_R S) \substack{\to\\\to\\\to} \cdots\right)$$

<u>Rmk:</u> In étale case, Weibel-Geller, McCarthy-Minasian

Idea: (Bhatt) First prove this for $L_{/\mathbb{F}_p}$) here use cofiber sequences for $L_{/\mathbb{F}_p}$

(if $A \to B \to C$ are maps of rings, cofiber seq $L_{B/A} \otimes^L_B C \to L_{C/A} \to L_{C/B}$)

& then bootstrap to $\Lambda^i L_{/\mathbb{F}_p}$

& show that this is a sheaf for flat topology.

& use HKR filtration $\rightsquigarrow$ $HH(\ /\mathbb{F}_p)$ is also sheaf for flat topology.

& $THH(R)/\sigma = HH(R/\mathbb{F}_p)$

$\rightsquigarrow THH(R)$ a sheaf.

$\rightsquigarrow TP(R)$ a sheaf (connectivity argument)

R smooth $\mathbb{F}_p$-alg.

Idea is to use sheaf property to understand $TP(R)$.

Construction $S = R_{perf}$
$$= \varinjlim_{x \mapsto x^p} R$$

Ex) $R = \mathbb{F}_p[x]$
$$R_{perf} = \mathbb{F}_p[x^{1/p^\infty}] = \bigcup_n \mathbb{F}_p[x^{1/p^n}].$$

Fact: R smooth $\mathbb{F}_p$ alg,
$R \to R_{perf} = S$ is faithfully flat.

To describe $TP(R)$, it suffices to describe
$TP(S)$, $TP(S \otimes_R S)$, ...
Idea is that these are easier (even degrees).

First case: $TP(S)$ S perfect $\mathbb{F}_p$-alg.

Prop $TP(S)_* = W(S)[x^{\pm 1}]$, $|x| = -2$.

Pf: $THH(S)_* = S[\sigma]$, $|\sigma| = 2$.
Run the S^1-Tate spectral sequence.
Identify the extension.

Filtration on $TP(S)$ is the Postnikov filtration! (crystalline coh of S is $W(S)$).

Unfortunately, this is not sufficient,
$S \otimes_R S$, $S \otimes_R S \otimes_R S$, ...

$S \otimes_R S$ is not perfect.

Ex) $R = \mathbb{F}_p[x]$, $S = \mathbb{F}_p[x^{1/p^\infty}]$.

$$S \otimes_R S \doteq \frac{\mathbb{F}_p[x_1^{1/p^\infty}, x_2^{1/p^\infty}]}{(x_1 = x_2)}$$

This is no longer a perfect ring!

$$(x_1^{1/p} - x_2^{1/p})^p = 0$$

$S \otimes_R S$ is a "<u>regular semiperfect</u>" ring, but the quotient of a perfect ring by a regular sequence.

More generally, all tensor products $S \otimes_R S \otimes_R \cdots \otimes_R S$

are regular semiperfect.

Needs to understand

1) $TP(T)_*$ T regular semiperfect

2) (derived) crystalline cohomology of T.

Main thm: (up to a mild completion), derived crystalline coh of T is a discrete ring, $TP(T)_* =$ 2-periodic derived crystalline cohomology.

$\mathbb{P}$

Def The BMS filtration on

$$F^i TP(R) = \mathrm{Tot}\left(\tau_{\le 2i} TP(S) \rightrightarrows \tau_{\le 2i} TP(S \otimes_R S) \substack{\to\\\to\\\to} \cdots\right)$$

This gives a filtration on $TP(R)$ & the fact that its associated is crystalline coh follows from the analogous assertion on $TP(S^{\otimes_R i})$

How do we work w/ regular semiperfect rings T?

Claim is that TP, ... are all in even degrees $\rightsquigarrow$ this all boils down $L_{T/\mathbb{F}_p}$.

<u>Prop</u> If T is regular semiperfect, $L_{T/\mathbb{F}_p}$ is suspension of a f.g. projective T-module. (in degree 1).

<u>Cor</u> T regular semiperfect, $HH(T/\mathbb{F}_p)_*$, $THH(T)_*$ are in even degrees.

Pf: Consequence ~~of~~ HKR filtration
$HH(T/\mathbb{F}_p)$ is filtered by

$$gr^i = \left(\Lambda^i_T L_{T/\mathbb{F}_p}\right)[i]$$

$L_{T/\mathbb{F}_p} \simeq P[1]$ P f.g. proj T-module

$$\mathrm{gr}^i = \left(\Lambda^i_T P[1]\right)[i]$$

$\wr\|$ Illusie

$$\left(\Gamma^i_T(P)[i]\right)[i]$$

$$\simeq \Gamma^i_T(P)[2i]$$ ← in even degrees.

$\rightsquigarrow$

$$HH_*(T/\mathbb{F}_p) \simeq \Gamma^*_T(P)$$ ← P has degree 2.

This also implies $THH(T)_*$ in even degrees,

$TP(T)_*$ is even.

Want to say explicitly what $TP(T)_*$ is, & relate to derived crystalline cohomology.

<u>Theorem</u> (BMS): T regular semiperfect ring, then $L\mathrm{crys}(T)$ (derived crystalline coh) is a discrete ring, in degree 0, & $A_{\mathrm{crys}}(T)$. Moreover, $TP_0(T)$ is the completion of $A_{\mathrm{crys}}(T)$ w.r.t. Nygaard filtration.

$\longrightarrow$ leads to BMS filtration.

What is $A_{crys}(T)$?

(e.g. $T = \mathbb{F}_p[x^{1/p^\infty}]/x$).

Construction of A_{crys}...

$$\begin{array}{ccc} T^{perf} & \longrightarrow & T \\ \Vert & & \\ \varprojlim_{x \mapsto x^p} T & & \end{array}$$

Ex) $\left(\mathbb{F}_p[x^{1/p^\infty}]/x \overset{(\ )^{perf}}{\rightsquigarrow} \mathbb{F}_p[x^{1/p^\infty}]^\wedge\right)$

Consider $W(T^{perf}) \xrightarrow{\theta} T$.

θ is the universal pro-nilpotent (p-adically complete) thickening of T.

if

$$\begin{tikzcd} S \arrow[r] & T \\ & W(T^{perf}) \arrow[u, "\theta"'] \arrow[ul, dotted] \end{tikzcd}$$

which is a nilpotent thickening

Def

$A_{crys}(T)$ = divided power envelope of θ

$\hookrightarrow W(T^{perf})$, add all divided powers of elements of kernel of θ.

(e.g. $x^i/i!$, $x \in \ker \theta$.
compatible w/ divided powers on p).

$A_{\mathrm{crys}}(T)$ is the universal (pro) PD thickening of T,

in the sense if

$$\begin{tikzcd} & S \arrow[d, "\text{nilpotent thickening w/ divided powers on ker}"] \\ A_{\mathrm{crys}}(T) \arrow[r] \arrow[ur, dashed] & T \end{tikzcd}$$

Consider

$$TP(T)_* \simeq \hat{A}_{\mathrm{crys}}(T)$$

e.g. in degree 0,

$$\begin{tikzcd} TP(T)_0 = \pi_0 THH(T)^{hS^1} \arrow[d] \arrow[dr, bend right] & \\ \pi_0 THH(T) \simeq T & \end{tikzcd}$$

"quasi syntomic site"

$$\underbrace{\mathbb{Z}_p[T]^\wedge}_{= R} \longrightarrow \underbrace{\widehat{\mathbb{Z}}_p[T^{1/p^\infty}]^\wedge}_{= S}$$