

# HOMOLOGICAL STABILITY MINICOURSE

## LECTURE 2: HOMOLOGICAL STABILITY FOR SYMMETRIC GROUPS

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ABSTRACT. In the second lecture of this minicourse, we prove homological stability for the symmetric groups  $\Sigma_n$ .

### 1. THE STRATEGY

In this lecture we will prove, following the strategy in [RWW17, Section 3] which originally goes back to Quillen:<sup>1</sup>

**Theorem 1.1** ([Nak60]). *The sequence  $B\Sigma_0 \xrightarrow{\sigma} B\Sigma_1 \xrightarrow{\sigma} B\Sigma_2 \xrightarrow{\sigma} \dots$  exhibits homological stability. More precisely,*

$$\sigma_* : H_*(B\Sigma_n; \mathbb{Z}) \longrightarrow H_*(B\Sigma_{n+1}; \mathbb{Z})$$

*is surjective if  $* \leq \frac{n}{2}$  and an isomorphism if  $* \leq \frac{n-1}{2}$ .*

Recall that we constructed  $BG$  as the geometric realisation of the nerve of a category  $* // G$ . As the notation suggests, this can be interpreted as a quotient, or more precisely a *homotopy quotient*. One can construct the homotopy quotient  $X // G$  of any space  $X$  with  $G$ -action by a group  $G$ , and here we just take  $X = *$ . By abuse of notation  $* // G = |N(* // G)|$ .<sup>2</sup> A reference for its construction and properties is [Rie14], but we will only need the following facts:

- (1) *Homotopy quotients are natural.* If  $X \rightarrow Y$  is an equivariant map between  $G$ -spaces then there is an induced map  $X // G \rightarrow Y // G$ .
- (2) *Homotopy quotients preserve homological connectivity.* If  $X \rightarrow Y$  is an equivariant map between  $G$ -spaces which is homologically  $d$ -connected then  $X // G \rightarrow Y // G$  is also homologically  $d$ -connected. (Recall that a map is *homologically  $d$ -connected* if it is an isomorphism on  $H_i$  for  $i < d$  and surjection on  $H_d$ .)
- (3) *Homotopy quotients commute with geometric realisation.* If  $X_\bullet$  is a semi-simplicial  $G$ -space, then  $||X_\bullet|| // G \simeq ||X_\bullet // G||$ . (We will explain the terminology and notation later.)
- (4) *Homotopy quotients of transitive  $G$ -sets.* If  $S$  is a transitive  $G$ -set, then  $S // G \simeq B\text{Stab}_G(s)$  for any  $s \in S$ .

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<sup>1</sup>Quillen used it to study the homology of general linear groups over finite fields. His argument was not published by Quillen, but appears in his notebooks [Qui74]. Unfortunately, the first few pages were left in the sun and were bleached. The argument was reconstructed and generalised in [SW20].

<sup>2</sup>This reflects that in modern homotopy theory, one does not really make a distinction between a category, its nerve, and the geometric realisation of its nerve.

When proving Theorem 1.1 we are only interested in the homology of  $B\Sigma_n$  in a range, so by (1) and (2) we may replace  $*$  with the trivial  $\Sigma_n$ -action by a different  $\Sigma_n$ -space  $X$  as long as  $X$  is homologically highly-connected. As (3) and (4) suggest, our desired  $X$  is of the form  $\|X_\bullet\|$  with each  $X_k$  a transitive  $G$ -set. Why is this a good idea? A geometric realisation  $\|X_\bullet\|$  comes with a canonical filtration, yielding a spectral sequence that will relate the homology of  $B\Sigma_n$  to that of the classifying spaces  $B\Sigma_{n-p}$  of various stabiliser groups. This will allow for an inductive argument.

## 2. INJECTIVE WORDS

Recall that  $\Delta$  is the category whose objects are non-empty ordered finite sets and whose morphisms are order-preserving maps; this has a skeleton given by the ordered finite sets  $[p] = (0 < \dots < p)$  for  $p \geq 0$ . The combinatorics of this category encodes various maps between the standard simplices  $\Delta^p = \{(t_0, \dots, t_p) \in [0, 1]^n \mid t_0 + \dots + t_p = 1\}$ : we can construct a functor

$$\Delta^\bullet: \Delta \longrightarrow \mathbf{Top},$$

which sends the morphism  $\delta_i: [p-1] \rightarrow [p]$  for  $0 \leq i \leq p$  which skips the  $i$ th element to the affine-linear inclusion  $\Delta^{p-1} \rightarrow \Delta^p$  opposite the  $i$ th vertex, and the morphism  $\sigma_j: [p] \rightarrow [p-1]$  for  $0 \leq j \leq p-1$  which doubles the  $j$ th element to the affine-linear surjection  $\Delta^p \rightarrow \Delta^{p-1}$  which collapses to the  $j$ th and  $(j+1)$ st vertices to same point. These morphisms generate all morphisms in the category  $\Delta$ .

A *simplicial set* is a functor  $\Delta^{\text{op}} \rightarrow \mathbf{Set}$  and a *simplicial space* is a functor  $\Delta^{\text{op}} \rightarrow \mathbf{Top}$ . The value  $X_p := X([p])$  is called the *space of  $p$ -simplices*. The morphism  $\delta_i$  induces a *face map*  $d_i: X_p \rightarrow X_{p-1}$ , and the morphism  $\sigma_i$  a *degeneracy map*  $s_j: X_{p-1} \rightarrow X_p$ . The geometric realisation is the coend

$$\|X_\bullet\| := \Delta^\bullet \otimes_{\Delta} X_\bullet = \left( \bigsqcup_{p \geq 0} \Delta^p \times X_p \right) / \sim$$

with  $\sim$  the equivalence relation generated by  $(\delta_i t, x) \sim (t, d_i x)$  and  $(\sigma_i t, x) \sim (t, s_i x)$ .

*Example 2.1.* In the previous lecture, we already used the simplicial set  $NC$  for a category  $C$  (we took the groupoid  $* // G$ ). Interpreting  $[p]$  has a poset, which is a particular type of category, we get a functor  $[\bullet]: \Delta \rightarrow \mathbf{Cat}$ . Then  $NC_p = \text{Hom}_C([p], C)$ , or more concretely,  $NC_p$  is the set of composable sequences of morphisms

$$C_0 \xrightarrow{f_0} C_1 \xrightarrow{f_1} \dots \xrightarrow{f_p} C_p.$$

The face maps are induced by precomposition, and explicitly given by composing or forgetting morphisms. The degeneracy maps are also induced by precomposition, and explicitly given by inserting identity morphisms.

It will suffice for our purposes to keep track of less structure, and replace  $\Delta$  by its subcategory  $\Delta_{\text{inj}}$  with the same objects but morphisms only injective order-preserving maps. A *semi-simplicial set* is a functor  $\Delta_{\text{inj}}^{\text{op}} \rightarrow \mathbf{Set}$  and a *semi-simplicial space* is a functor  $\Delta_{\text{inj}}^{\text{op}} \rightarrow \mathbf{Top}$ . That, it is the analogue of a simplicial space which only face maps and no degeneracy maps. We can restrict  $\Delta^\bullet$  to  $\Delta_{\text{inj}}$  and once more take the coend to get a geometric realisation

$$\|X_\bullet\| := \Delta^\bullet \otimes_{\Delta_{\text{inj}}} X_\bullet = \left( \bigsqcup_{p \geq 0} \Delta^p \times X_p \right) / \sim$$

with  $\sim$  the equivalence relation generated by  $(\delta_i t, x) \sim (t, d_i x)$ . A reference for semi-simplicial spaces and their properties is [ERW19].

*Remark 2.2.* Under suitable cofibrancy conditions, for a simplicial space  $X_\bullet$  we have  $|X_\bullet| \simeq \text{hocolim}_{\Delta^{\text{op}}} X_\bullet$ . One advantage for semi-simplicial spaces, is that for a semi-simplicial space  $X_\bullet$  we *always* have  $\|X_\bullet\| \simeq \text{hocolim}_{\Delta_{\text{inj}}^{\text{op}}} X_\bullet$ . This is a consequence of [ERW19, Theorem 2.2], as this theorem implies cofibrantly replacing  $X_\bullet$  does not affect the weak homotopy type of the geometric realisation.

We now define the semi-simplicial set with  $\Sigma_n$ -action which will replace  $*$ . Let  $\text{FI}$  be the category whose objects are finite sets and whose morphisms are injections.

**Definition 2.3.**  $W_n(\underline{1})_\bullet$  is the semi-simplicial set with  $p$ -simplices given by

$$W_n(\underline{1})_\bullet = \text{Hom}_{\text{FI}}([p], \underline{n})$$

and face maps  $d_i$  given by precomposition with  $\delta_i: [p-1] \rightarrow [p]$ .

That is,  $W_n(\underline{1})_p$  has as  $p$ -simplices the ordered words  $(m_0 m_1 \cdots m_p)$  of elements of  $\underline{n}$  and no letter duplicated. The  $i$ th face map forgets the  $i$ th letter  $m_i$ . This explains why we call this the *semi-simplicial set of injective words*. The notation is rather complicated, but will become clear in the next lecture.

*Example 2.4.*  $W_2(\underline{1})_\bullet$  has:

- two 0-simplices given by the words (1) and (2),
- two 1-simplices given by the words (12) and (21).

Its geometric realisation is a circle.

*Example 2.5.*  $W_3(\underline{1})_\bullet$  has:

- three 0-simplices (1), (2), (3),
- six 1-simplices (12), (21), (13), (31), (23), (32),
- six 2-simplices (123), (213), (132), (312), (231), (321).

We will see this is 1-connected. Let us just pick the loop given by the 1-simplices corresponding to (12) and (21) and see this null-homotopic: it is the boundary obtained when we glue the 2-simplices corresponding to (123) and (213) along their common 1-simplices.

The group  $\Sigma_n$  acts on  $W_n(\underline{1})_\bullet$  by post-composition, and hence on the geometric realisation. We have that:

**Proposition 2.6.**

- (i)  $\|W_n(\underline{1})_\bullet\|$  is homologically  $\frac{n-1}{2}$ -connected.
- (ii)  $W_n(\underline{1})_p$  is a transitive  $\Sigma_n$ -set, and the stabiliser of  $x \in W_n(\underline{1})_p$  is the group of permutations of  $\underline{n} \setminus \text{im}(x)$ .

Here (ii) is evident, but (i) requires a proof which we postpone to the next lecture. The upshot is that  $\|W_n(\underline{1})_\bullet\| // \Sigma_n$  can serve a replacement for  $* // \Sigma_n$  for computing the homology in a range.

### 3. THE GEOMETRIC REALISATION SPECTRAL SEQUENCE

A filtration  $F_0 X \subset F_1 X \subset \dots$  on a space  $X$  makes the singular chains  $C_*(X)$  into a filtered chain complex by setting  $F_r C_*(X) := \text{im}(C_*(F_r X) \rightarrow C_*(X))$ . Assuming

that  $F_{r-1}X \rightarrow F_r X$  is a cofibration and  $F_r X/F_{r-1}X$  is at least  $(r-1)$ -connected, this gives a strongly-convergent first-quadrant *spectral sequence*

$$E_{p,q}^1 = \tilde{H}_{p+q}(F_p X/F_{p-1}X; \mathbb{Z}) \implies H_{p+q}(X; \mathbb{Z})$$

with differentials given by  $d^r: E_{p,q}^r \rightarrow E_{p-r,q+r-1}^r$ .

*Remark 3.1.* If you are unfamiliar with these notions, I recommend you look at [McC01, Hat]. Roughly, a spectral sequence is an algebraic object that conveniently packages all long exact sequences in homology for the pairs  $(F_s X, F_r X)$  with  $s \geq r$  with the goal of compute the homology of  $X$ .

We can in particular apply this to the geometric realisation  $\|X_\bullet\|$ . This has a filtration

$$F_r \|X_\bullet\| := \left( \bigsqcup_{0 \leq p \leq r} \Delta^p \times X_p \right) / \sim$$

with equivalence relation  $\sim$  as before, all of whose maps are cofibrations under a mild condition on  $X_\bullet$  that will be satisfied in examples in these notes. The associated graded is given by

$$\frac{F_r \|X_\bullet\|}{F_{r-1} \|X_\bullet\|} = \frac{\Delta^r}{\partial \Delta^r} \wedge (X_r)_+ \quad (1)$$

so is at least  $(r-1)$ -connected. Thus we get [Seg68] (see also [ERW19, Section 1.4]):

**Theorem 3.2.** *There is a strongly convergent first quadrant spectral sequence*

$$E_{p,q}^1 = H_q(X_p; \mathbb{Z}) \implies H_{p+q}(\|X_\bullet\|; \mathbb{Z})$$

with differentials given by  $d^r: E_{p,q}^r \rightarrow E_{p-r,q+r-1}^r$ . Moreover  $d^1: E_{p,q}^1 \rightarrow E_{p-1,q}^1$  is given by  $\sum_{i=0}^p (-1)^i (d_i)_*$ , and the edge homomorphism  $E_{0,q}^1 \rightarrow E_{0,q}^\infty \rightarrow H_q(X; \mathbb{Z})$  is equal to the map induced on homology by the inclusion  $X_0 \rightarrow \|X_\bullet\|$ .

*Proof.* The identification of the  $E^1$ -page follows from (1) and the suspension isomorphism. The description of the abutment and edge homomorphism is as for any spectral sequence of a filtered space. For the identification of the  $d^1$ -differential, see the references.  $\square$

#### 4. THE PROOF OF THEOREM 1.1

We have now gathered all ingredients for the proof of Theorem 1.1. It is a proof by strong induction, so we assume we have proven the result for  $m \leq n$  and we will prove it for  $n+1$ . There is nothing to prove when  $n+1 \leq 2$ , so we may assume that  $n+1 \geq 3$ .

**Step 1: Replacing  $*$  by  $\|W_{n+1}(\underline{1})_\bullet\|$ .** By Proposition 2.6,  $\|W_{n+1}(\underline{1})_\bullet\|$  is homologically  $\frac{n}{2}$ -connected so the map  $\|W_{n+1}(\underline{1})_\bullet\| \rightarrow *$  is  $(\frac{n}{2}+1)$ -connected. Taking homotopy quotients by  $\Sigma_{n+1}$ , (1) yields a map

$$\|W_{n+1}(\underline{1})_\bullet\| // \Sigma_{n+1} \longrightarrow * // \Sigma_{n+1}$$

which is homologically  $(\frac{n}{2}+1)$ -connected by (2). To prove Theorem 1.1, we may thus replace  $* // \Sigma_n$  by  $\|W_{n+1}(\underline{1})_\bullet\| // \Sigma_{n+1}$ .

**Step 2: The  $E^1$ -page of the geometric realisation spectral sequence.** Next, (3) provides a weak homotopy equivalence

$$\|W_{n+1}(\underline{1})\bullet\| // \Sigma_{n+1} \simeq \|W_{n+1}(\underline{1})\bullet\| // \Sigma_{n+1}\|.$$

Then Theorem 3.2 provides a spectral sequence

$$E_{p,q}^1 = H_q(W_{n+1}(\underline{1})_p // \Sigma_{n+1}; \mathbb{Z}) \implies H_{p+q}(X; \mathbb{Z}).$$

Let us now identify the  $E^1$ -page more explicitly, as well as the  $d^1$ -differential. Since  $W_{n+1}(\underline{1})_p$  is a transitive  $\Sigma_{n+1}$ , for any injective map  $f: [p] \rightarrow \underline{n+1}$ , (4) says that the map

$$* // \text{Stab}_{\Sigma_{n+1}}(f) \longrightarrow W_{n+1}(\underline{1})_p // \Sigma_{n+1}$$

sending  $*$  to  $f$  is a weak homotopy equivalence. Taking  $f$  to be the inclusion  $\iota_p$  of the last  $p+1$  elements,  $\text{Stab}_{\Sigma_{n+1}}(\iota_p) \hookrightarrow \Sigma_{n+1}$  is the usual inclusion  $\Sigma_{n-p} \hookrightarrow \Sigma_{n+1}$  as acting on the first  $n-p$  elements. This allows us to make the identification

$$E_{p,q}^1 \cong H_q(B\Sigma_{n-p}; \mathbb{Z}), \quad (2)$$

but it is important to recall how this identification is made when we next compute the  $d^1$ -differential.

Indeed,  $d^1 = \sum (-1)^i (d_i)_*$  and even though the diagram

$$\begin{array}{ccc} * // \text{Stab}_{\Sigma_{n+1}}(f) & \xrightarrow{\cong} & W_{n+1}(\underline{1})_p // \Sigma_{n+1} \\ \downarrow & & \downarrow d_i // \Sigma_{n+1} \\ * // \text{Stab}_{\Sigma_{n+1}}(d_i f) & \xrightarrow{\cong} & W_{n+1}(\underline{1})_{p-1} // \Sigma_{n+1} \end{array}$$

commutes it is not true in general that  $d_i \iota_p = \iota_{p-1}$ . Rather, we have  $h_i d_i \iota_p = \iota_{p-1}$  where  $h_i$  is an element of  $\Sigma_{n+1}$  that sends  $d_i \iota_p$  to  $\iota_{p-1}$  (there is more than one such element). The correct commuting diagram involving only standard inclusions is

$$\begin{array}{ccc} * // \text{Stab}_{\Sigma_{n+1}}(\iota_p) & \xrightarrow{\cong} & W_{n+1}(\underline{1})_p // \Sigma_{n+1} \\ \downarrow & & \downarrow d_i // \Sigma_{n+1} \\ * // \text{Stab}_{\Sigma_{n+1}}(d_i \iota_p) & \xrightarrow{\cong} & W_{n+1}(\underline{1})_{p-1} // \Sigma_{n+1} \\ c_{h_i} \downarrow & & \simeq \downarrow c'_{h_i} \\ * // \text{Stab}_{\Sigma_{n+1}}(\iota_{p-1}) & \xrightarrow{\cong} & W_{n+1}(\underline{1})_{p-1} // \Sigma_{n+1} \end{array}$$

where  $c_{h_i}: \text{Stab}_{\Sigma_{n+1}}(d_i \iota_p) \rightarrow \text{Stab}_{\Sigma_{n+1}}(\iota_{p-1})$  is induced by conjugation in  $\Sigma_{n+1}$  with  $h_i$ , and  $c'_{h_i}$  is induced by the map of  $\Sigma_{n+1}$ -sets  $W_{n+1}(\underline{1})_{p-1} \rightarrow W_{n+1}(\underline{1})_{p-1}$  sending given by multiplication with  $h_i$  (one also needs to then twist the action by conjugation by  $h_i$ ). The map  $c'_{h_i}$  is homotopic to the identity, so the upshot is that the following diagram commutes

$$\begin{array}{ccc} H_*(B\Sigma_{n-p}; \mathbb{Z}) & \xrightarrow[\cong]{\iota_p} & H_*(W_{n+1}(\underline{1})_p // \Sigma_{n+1}; \mathbb{Z}) \\ (c_{h_i} \circ \text{inc})_* \downarrow & & \downarrow (d_i)_* \\ H_*(B\Sigma_{n-p+1}; \mathbb{Z}) & \xrightarrow[\cong]{\iota_{p+1}} & H_*(W_{n+1}(\underline{1})_{p-1} // \Sigma_{n+1}; \mathbb{Z}). \end{array}$$

We are free to choose  $h_i$ , and we shall make a fortunate choice: we take it to be the transposition swapping  $n-p+1$  and  $n-p+i$ . This has the advantage of commuting

with the image of  $\Sigma_{n-p}$ , so that the left map is equal to just  $\sigma_*$ . The upshot is that under the identification of (2), we can identify the  $d^1$ -differential as

$$d^1 = \sum_{i=0}^p (-1)^i \sigma_* = \begin{cases} \sigma_* & \text{if } p > 0 \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{array}{ccccccc} & \vdots & & \vdots & & \vdots & & \vdots \\ 4 & \cdots & H_4(B\Sigma_n) & \xleftarrow{0} & H_4(B\Sigma_{n-1}) & \xleftarrow{\sigma} & H_4(B\Sigma_{n-2}) & \xleftarrow{0} & H_4(B\Sigma_{n-3}) \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ 3 & \cdots & H_3(B\Sigma_n) & \xleftarrow{0} & H_3(B\Sigma_{n-1}) & \xleftarrow{\sigma} & H_3(B\Sigma_{n-2}) & \xleftarrow{0} & H_3(B\Sigma_{n-3}) \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ 2 & \cdots & H_2(B\Sigma_n) & \xleftarrow{0} & H_2(B\Sigma_{n-1}) & \xleftarrow{\sigma} & H_2(B\Sigma_{n-2}) & \xleftarrow{0} & H_2(B\Sigma_{n-3}) \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ 1 & \cdots & H_1(B\Sigma_n) & \xleftarrow{0} & H_1(B\Sigma_{n-1}) & \xleftarrow{\sigma} & H_1(B\Sigma_{n-2}) & \xleftarrow{0} & H_1(B\Sigma_{n-3}) \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ 0 & \cdots & H_0(B\Sigma_n) & \xleftarrow{0} & H_0(B\Sigma_{n-1}) & \xleftarrow{\sigma} & H_0(B\Sigma_{n-2}) & \xleftarrow{0} & H_0(B\Sigma_{n-3}) \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ & & 0 & & 1 & & 2 & & 3 \end{array}$$

FIGURE 1. The  $E^1$ -page entries  $E_{p,q}^1$  and the  $d^1$ -differentials (we drop the coefficients from homology for the sake of readability).

**Step 3: A spectral sequence argument.** We want to show that the edge homomorphism

$$E_{0,q}^1 = H_q(B\Sigma_n; \mathbb{Z}) \longrightarrow H_q(\|W_{n+1}(1)\bullet\| \Sigma_{n+1}; \mathbb{Z}) \xrightarrow{\cong} H_q(B\Sigma_{n+1}; \mathbb{Z})$$

is a surjection for  $* \leq \frac{n}{2}$  and an isomorphism for  $* \leq \frac{n-1}{2}$ . Indeed, in this range the right map is an isomorphism (as indicated) and it is easy to identify the composition with the stabilisation map  $\sigma$ .

We will do so by studying the geometric realisation spectral sequence, whose  $E^1$ -page looks like Figure 1. By the inductive hypothesis, whenever we see a  $d^1$ -differential equal to  $\sigma$  it is surjective or even an isomorphism in a range.

Putting in the explicit ranges, we get that the  $E^2$ -page vanishes in a range for  $p > 0$ . There are two cases, (a)  $n = 2k + 1$  odd and (b)  $n = 2k + 1$  even:

- (a) The precise vanishing range is that for  $r \geq 0$ ,  $E_{2r+1,q}^2 = E_{2r+2,q}^2 = 0$  for  $q \leq k - r - 1$ . For example, if  $n = 9$  then  $E_{2,q}^1 = H_q(B\Sigma_7; \mathbb{Z}) \rightarrow H_q(B\Sigma_8; \mathbb{Z})$  is a surjection for  $* \leq \frac{7}{2}$  and an isomorphism for  $* \leq \frac{6}{2}$ . That is, two adjacent columns connected by a  $\sigma$  vanish in a range, the first pair from the left vanishing for  $q \leq k - 1$  and this range going down one degree whenever we move two columns to the right. See Fig. 2 for  $2k = 9$ .

Furthermore, the higher differentials are  $d^r: E_{p,q}^r \rightarrow E_{p-r,q+r-1}^r$  (so upwards and to the left). Thus no differential with non-zero domain can enter the entries  $E_{0,q}^2 = H_q(B\Sigma_{2k}; \mathbb{Z})$  for  $q \leq k$ , and that no further non-zero groups contribute to  $E_{0,q}^\infty$ . This implies that  $H_q(B\Sigma_{2k+1}; \mathbb{Z}) \rightarrow H_q(B\Sigma_{2k+2}; \mathbb{Z})$  is an isomorphism for  $q \leq k = \lfloor \frac{2k+1-1}{2} \rfloor = \lfloor \frac{2k+1}{2} \rfloor$ ; this what we needed to prove.

- (b) The precise vanishing range is that for  $r \geq 0$ ,  $E_{2r+1,q}^2 = 0$  and  $q \leq k - r - 1$  and  $E_{2r+2,q}^2 = 0$  for  $q - r - 2$ . This is the same pattern as in case (a), but in the right column in each pair the range is one lower. See Fig. 2 for  $2k = 8$ .

The result is that no differential with non-zero domain can enter the entries  $E_{0,q}^2 = H_q(B\Sigma_{2k}; \mathbb{Z})$  for  $q \leq k-1$ , and that no further non-zero groups contribute to  $E_{0,q}^\infty$ . Furthermore, a single  $d^2$ -differential with non-zero domain can enter the entry  $E_{0,k}^2 = H_k(B\Sigma_{2k}; \mathbb{Z})$  and again no further non-zero groups can contribute. This implies that  $H_q(B\Sigma_{2k}; \mathbb{Z}) \rightarrow H_q(B\Sigma_{2k+1}; \mathbb{Z})$  is an isomorphism for  $q \leq k-1 = \lfloor \frac{2k-1}{2} \rfloor$  and a surjection for  $q = \lfloor \frac{2k}{2} \rfloor = k$ ; this what we needed to prove.

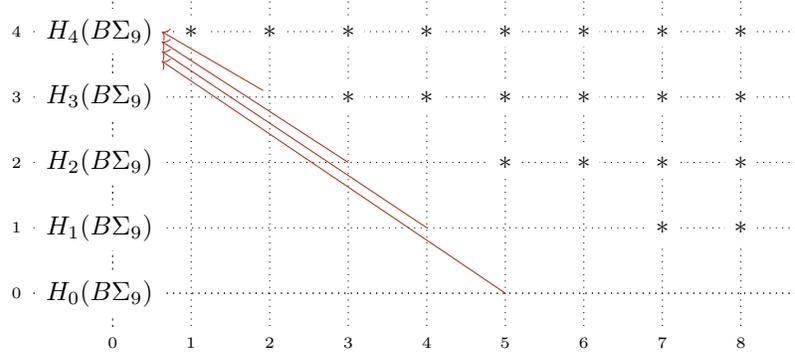


FIGURE 2. The  $E^2$ -page entries  $E_{p,q}^2$  for  $n = 2k + 1 = 9$ . The empty entries are 0, the \* means unknown groups. We have drawn the higher differentials into the entry  $E_{0,4}^2 = H_4(B\Sigma_9; \mathbb{Z})$ .

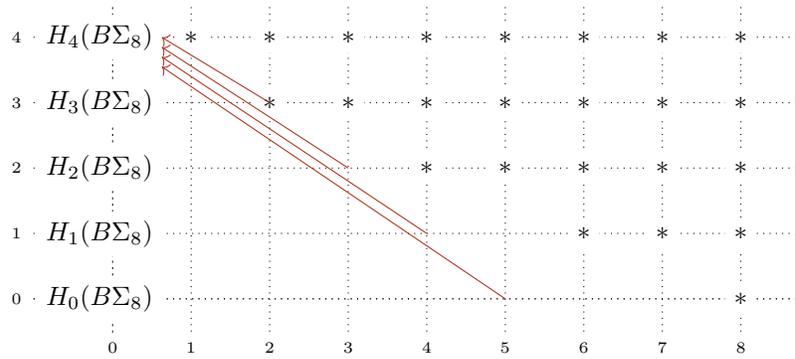


FIGURE 3. The  $E^2$ -page entries  $E_{p,q}^2$  for  $n = 2k = 8$ . The empty entries are 0, the \* means unknown groups. We have drawn the higher differentials into the entry  $E_{0,4}^2 = H_4(B\Sigma_8; \mathbb{Z})$ .

## 5. EXERCISES

*Exercise 5.1* (Simplicial and semi-simplicial spaces).

- (i) Let  $\mathbf{sTop}$  and  $\mathbf{ssTop}$  denote the categories of simplicial, resp. semi-simplicial, spaces. Prove that the forgetful functor  $U: \mathbf{sTop} \rightarrow \mathbf{ssTop}$  has a left-adjoint  $F$ . (Hint: it is given by “freely adjoining degeneracies”.)
- (ii) Prove that  $|F(X_\bullet)| \cong ||X_\bullet||$  for  $X_\bullet \in \mathbf{ssTop}$ .

*Exercise 5.2* (The fundamental group of  $W_n(1)_\bullet$ ). Use Seifert–van Kampen to prove that  $\|W_n(1)_\bullet\|$  is 1-connected for  $n \geq 3$ .

*Exercise 5.3* (Homotopy quotients preserve connectivity). There is a fibration sequence natural in  $G$ -spaces  $X$

$$X \longrightarrow X // G \longrightarrow * // G.$$

Use the associated Serre spectral sequence to prove property (2) of homotopy quotients.

*Exercise 5.4* (Homotopy quotients of  $G$ -sets). If  $X$  is a set with  $G$ -action, then the homotopy quotient  $X // G$  has a model as the geometric realisation of the nerve of the category with objects elements  $x \in X$  and a unique morphism from  $x$  to  $gx$ . We also denote this category by  $X // G$ .

- (i) For  $x \in X$  construct a functor  $\iota_x: * // \text{Stab}_G(x) \rightarrow X // G$  sending  $*$  to  $x$ .
- (ii) Prove that  $\iota_x$  is an equivalence of categories if  $X$  is a transitive  $G$ -set and conclude that the map induced by  $\iota_x$  on geometric realisation of nerves is a homotopy equivalence. This is property (4) of homotopy quotients.
- (iii) For  $h \in G$  construct a functor  $c_h: X // G \rightarrow X // G$  sending  $x$  to  $hx$ .
- (iv) Prove that there is a natural transformation  $\text{id} \Rightarrow c_h$  and conclude that the map induced by  $c_h$  on geometric realisation of nerves is homotopic to the identity. This justifies a claim in Step 2 of Section 4.
- (v) Can the homotopy in (iv) be taken to be based?

*Exercise 5.5* (Homological stability with constant coefficients). Prove that Theorem 3.2 goes through when we replace the coefficients  $\mathbb{Z}$  with another abelian group  $A$ .

*Exercise 5.6* (Homological stability with abelian coefficients). If  $M$  be a  $\mathbb{Z}/2$ -module, then we can use the sign homomorphism  $\text{sign}: \Sigma_n \rightarrow \mathbb{Z}/2$  to make it into a local coefficient system  $M_n$  on  $B\Sigma_n$  and define  $H_*(B\Sigma_n; M_n)$ .

- (i) Prove that the pullback of  $M_{n+1}$  along  $\sigma: B\Sigma_n \rightarrow B\Sigma_{n+1}$  is canonically isomorphic to  $M_n$ . Construct a stabilisation map

$$\sigma_*: H_*(B\Sigma_n; M_n) \longrightarrow H_*(B\Sigma_{n+1}; M_{n+1}).$$

- (ii) Modify the argument in this lecture to prove that that  $\sigma_*: H_*(B\Sigma_n; M_n) \rightarrow H_*(B\Sigma_{n+1}; M_{n+1})$  is a surjection if  $* \leq \frac{n-1}{3}$  and an isomorphism if  $* \leq \frac{n-3}{3}$ . (Hint:  $(c_{h_i} \circ \text{inc})_*$  is no longer equal to  $\sigma_*$ , but it is so on the image of  $H_*(B\Sigma_{n-p-1}; M_{n-p-1})$  in  $H_*(B\Sigma_{n-p}; M_{n-p})$ .)

*Exercise 5.7* (Homological stability for alternating groups). By picking  $M$  appropriately and invoking Shapiro’s lemma, use Exercise 5.6 to deduce homological stability for alternating groups.

*Remark 5.8.* Homological stability for alternating groups goes back to Mann [Man85].

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