

HOMOLOGICAL STABILITY MINICOURSE
LECTURE 3: HOMOLOGICAL STABILITY FOR AUTOMORPHISM
GROUPS

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ABSTRACT. In the third lecture of this minicourse, we finish the proof of homological stability for symmetric groups by proving that the semi-simplicial set of injective words is highly-connected. We then explain the general framework for homological stability due to Randal-Williams and Wahl, with an application to a result of Dwyer.

1. THE SEMI-SIMPLICIAL SET OF INJECTIVE WORDS IS HIGHLY-CONNECTED

Last lecture, we proved that the sequence $B\Sigma_0 \xrightarrow{\sigma} B\Sigma_1 \xrightarrow{\sigma} B\Sigma_2 \xrightarrow{\sigma} \cdots$ exhibits homological stability: more precisely,

$$\sigma_* : H_*(B\Sigma_n; \mathbb{Z}) \longrightarrow H_*(B\Sigma_{n+1}; \mathbb{Z})$$

is surjective if $* \leq \frac{n}{2}$ and an isomorphism if $* \leq \frac{n-1}{2}$. We gave a complete proof, with the exception of using the following result as input:

Proposition 1.1. $\|W_n(\underline{1})_\bullet\|$ is homologically $\frac{n-1}{2}$ -connected.

Remark 1.2. It is in fact known to be $(n-2)$ -connected (e.g. [Far79, Ker05, RW13a, Gan17]), but we will not use this. See Exercise 4.2 for a proof which uses a technique which is useful for proving other semi-simplicial sets are highly-connected.

I want to prove this, after recalling the definition of $W_n(\underline{1})_\bullet$, because we will soon see that the connectivity of semi-simplicial sets like $W_n(\underline{1})$ is often the crux for proving homological stability results. Recall that \mathbf{FI} is the category of finite sets and injections.

Definition 1.3. $W_n(\underline{1})_\bullet$ is the semi-simplicial set with p -simplices given by

$$W_n(\underline{1})_\bullet = \mathrm{Hom}_{\mathbf{FI}}([p], \underline{n})$$

and face maps d_i given by precomposition with $\delta_i : [p-1] \rightarrow [p]$.

Proof of Proposition 1.1. We will give a proof due to Randal-Williams [RW]. The proof is by strong induction over n ; we did the cases $n \leq 2$ before and the case $n = 3$ was an exercise. So we may assume $n \geq 4$, and suppose the cases $< n$ to be known. In our induction we will use different subsets of \underline{n} , so it is convenient to temporarily write $W(S)_\bullet$ for the semi-simplicial set obtained by replacing \underline{n} with S .

The inclusions $\underline{n} \setminus \{i\} =: S_i \hookrightarrow \underline{n}$ for $1 \leq i \leq n$ of all $(n-1)$ -element subsets give a cover by subcomplexes of the $(n-2)$ -skeleton of $\|W(\underline{n})_\bullet\|$ with its canonical CW-structure. Thus the map

$$\cup_{i=1}^n \|W(S_i)_\bullet\| \longrightarrow \|W(\underline{n})_\bullet\|$$

induces an surjection on H_* for $* \leq n-2$, and $n-2 \geq \frac{n-1}{2}$ when $n \geq 4$.

We will now prove that $\bigoplus_{i=1}^n H_*(\|W(S_i)\bullet\|; \mathbb{Z}) \rightarrow H_*(\cup_{i=1}^n \|W(S_i)\bullet\|; \mathbb{Z})$ is surjective for $* \leq \frac{n-1}{2}$. To do so, we use the Mayer–Vietoris spectral sequence for this cover:

$$E_{pq}^1 = \bigoplus_{1 \leq i_0 < \dots < i_p \leq n} H_q(\cap_{j=1}^p \|W(S_{i_j})\bullet\|; \mathbb{Z}) \implies H_{p+q}(\cup_{i=1}^n \|W(S_i)\bullet\|; \mathbb{Z}).$$

Since $\cap_{j=0}^p \|W(S_{i_j})\bullet\| \cong \|W(\cap_{j=0}^p S_{i_j})\bullet\|$ is isomorphic to $\|W(n-p-1)\bullet\|$, the entries $E_{p,q}^1$ vanish for $0 < q \leq \frac{n-p-2}{2}$. Furthermore, the chain complex $(E_{p,0}^1, d^1)$ is the cellular chain complex of $\partial\Delta^{n-1}$ and hence on the E^2 -page, we get that the bottom row $E_{p,0}^2$ vanishes for $0 < p < n-2$. Since $n \geq 4$, this implies that the edge homomorphism is surjective for $* \leq \frac{n-1}{2}$, which proves the desired statement.

Now we observe that composition

$$E_{0,q}^1 \longrightarrow H_q(\cup_{i=1}^n \|W(S_i)\bullet\|; \mathbb{Z}) \xrightarrow{\cong} H_q(\|W(\underline{n})\bullet\|; \mathbb{Z})$$

is induced by the inclusions $\|W(S_i)\bullet\| \rightarrow \|W(\underline{n})\bullet\|$, which are null-homotopic by “coning off” using the element $i \in \underline{n}$ (see Exercise 4.1). Thus in degrees $0 < * \leq \frac{n-1}{2}$

$$H_*(\cup_{i=1}^n \|W(S_i)\bullet\|; \mathbb{Z}) \longrightarrow H_*(\|W(\underline{n})\bullet\|; \mathbb{Z})$$

is both zero and surjective, and hence the target vanishes. This completes the proof of the induction step. \square

Remark 1.4. From an exercise in the previous lecture, we know that $\|W_n(\underline{1})\bullet\|$ is also simply-connected for $n \geq 3$. Hence in that case, it is not just homologically $\frac{n-1}{2}$ -connected but actually $\frac{n-1}{2}$ -connected.

2. THE FRAMEWORK OF RANDAL–WILLIAMS AND WAHL

We did not use much about symmetric groups in our arguments and it is possible to completely formalise the properties that we did use; doing so yields [RWW17].

A *monoidal category* \mathbf{C} is a category with a functor

$$\oplus : \mathbf{C} \times \mathbf{C} \longrightarrow \mathbf{C}$$

and an object $\mathbb{1} \in \mathbf{C}$ serving as a unit. There are associativity and unitality isomorphisms relating these, see [ML98]. It is a *symmetric monoidal* if we are additionally given compatible natural isomorphisms $\beta_{X,Y} : X \oplus Y \rightarrow Y \oplus X$ so that $\beta_{X,Y} \circ \beta_{Y,X} = \text{id}$.

When \mathbf{G} is a symmetric monoidal groupoid, then given objects $A, X \in \mathbf{C}$ we can form the automorphism groups

$$G_n := \text{Aut}_{\mathbf{C}}(A \oplus X^{\oplus n}),$$

and the functor $- \oplus X$ induces a stabilisation map $\sigma : G_n \rightarrow G_{n+1}$ between these.

Question 2.1. When does the sequence $BG_0 \xrightarrow{\sigma} BG_1 \xrightarrow{\sigma} BG_2 \xrightarrow{\sigma} \dots$ exhibit homological stability?

Theorem 2.2 ([RWW17]). *Suppose that a symmetric monoidal groupoid \mathbf{G} has the following properties:*

- (i) *the monoid of isomorphism classes of objects of \mathbf{G} has cancellation,*
- (ii) *$\text{Aut}(B) \rightarrow \text{Aut}(B \oplus X)$ is injective for all B .*

Then we will momentarily describe semi-simplicial sets $W_n(A, X)_\bullet$ with the following property: if there is a $k \geq 2$ with $\|W_n(A, X)_\bullet\|$ homologically $\frac{n-2}{k}$ -connected for all $n \geq 2$, then

$$\sigma_*: H_*(BG_n; \mathbb{Z}) \longrightarrow H_*(BG_{n+1}; \mathbb{Z})$$

is a surjection for $* \leq \frac{n}{k}$ and an isomorphism for $* \leq \frac{n-1}{k}$.

Remark 2.3. More generally, [RWW17] allows braided monoidal groupoids as inputs. Moreover, the classifying space of a symmetric monoidal groupoid is an E_2 -algebra which is a disjoint union of Eilenberg–MacLane spaces. In [Kra19], Krannich proved the analogous result for general E_2 -algebras, allowing one to drop the cancellation and injectivity assumptions. This for example allows one to study diffeomorphism groups of high-dimensional manifolds. The reader may notice that the assumptions in [RWW17] are slightly different than those of Theorem 2.2. This is explained in [Kra19, Section 7.3].

The construction of $W_n(A, X)_\bullet$ uses a construction due to Quillen, which we denote $\mathcal{U}\mathcal{G}$ following [RWW17]: this is the category with the same objects as \mathcal{G} but morphisms from X to Y given by an equivalence class of pairs (Z, f) with $f: Z \oplus X \rightarrow Y$ a morphism in \mathcal{G} . Two of these, (Z, f) and (Z', f') , are equivalent if there is a morphism $g: Z \rightarrow Z'$ in \mathcal{G} such that the following diagram commutes:

$$\begin{array}{ccc} Z \oplus X & \xrightarrow{g \oplus \text{id}} & Z' \oplus X \\ & \searrow f & \swarrow f' \\ & & Y \end{array}$$

This inherits a symmetric monoidal structure. Then $W_n(X)_p = \text{Hom}_{\mathcal{U}\mathcal{G}}(X^{\oplus p}, A \oplus X^{\oplus n})$ and the i th face map is induced by precomposition.

Example 2.4. Suppose \mathcal{G} is the groupoid FB of finite sets and bijections, on which disjoint union gives a symmetric monoidal structure. Then $\mathcal{U}\mathcal{G}$ is the category FI of finite sets and injections. Taking $A = \emptyset$ and $X = \underline{1}$ we get that $G_n = \Sigma_n$ and $W_n(\emptyset, X)_\bullet = W_n(\underline{1})_\bullet$ as above. Thus Theorem 2.2 recovers Nakaoka’s theorem.

Example 2.5. Suppose that \mathcal{G} is the groupoid of finitely-generated abelian groups and their automorphisms, on which direct sum gives a symmetric monoidal structure. Then taking $A = \mathbb{Z}$, $X = \mathbb{Z}$ we get that $G_n = \text{GL}_{n+1}(\mathbb{Z})$. Properties (i) and (ii) hold by the classification of finitely-generated abelian groups. The semi-simplicial sets $W_n(\mathbb{Z}, \mathbb{Z})_\bullet$ were essentially studied by Charney [Cha84]; her proof can be adapted to prove that these are $\frac{n-2}{2}$ -connected (this is the hardest step in the argument). We conclude that the stabilisation map

$$\sigma: \text{BGL}_{n+1}(\mathbb{Z}) \longrightarrow \text{BGL}_{n+2}(\mathbb{Z})$$

induced by the inclusion $\text{GL}_{n+1}(\mathbb{Z}) \rightarrow \text{GL}_{n+2}(\mathbb{Z})$ in the top-left corner, induces a surjection on H_* for $* \leq \frac{n}{2}$ and an isomorphism for $* \leq \frac{n-1}{2}$.

Example 2.6. Here is an incomplete list of further sequences of groups whose classifying spaces exhibit homological stability and equally incomplete references (see also Section 5 of [RWW17]):

- *General linear groups of rings of finite stable rank* (this includes any ring you’re likely to think of) [vdK80, Cha80, Dwy80, Cha84, Bet86b, NS89, Gui89, Bet92, HT10, GKRW18, SW20, GKRW20].

- *Unitary groups of rings of finite unitary stable rank* (again this includes any ring you're likely to think of) [Fri76, Pan87, MvdK02, Mir05, Col11, Ess13, SW20]; this includes *symplectic groups* [vdKL11] and *orthogonal groups* [Vog79, Vog81, Vog82, Bet86a, Cha87, Bet87, Bet90, Cat07].
- *Automorphism groups of free groups* [Hat95, HV98a, HV98b, RW18] and related groups [HV04, HW05, HVW06, HW08, Zar14]
- *Diffeomorphism groups or mapping class groups of surfaces* [Har85, Har90, Iva93, Wah08, Bol12, Wah13, RW14, RW16, HV17, GKRW19] and *3-manifolds* [HW10, Lam15, Kup20]; this includes *braid groups* [Arn69, Fuk70, CLM76, Vai78]. The latter is an example of a configuration space, which we will discuss in more detail in the next lecture.
- *Diffeomorphism groups of high-dimensional manifolds* [Per16a, Per16b, GRW17, Per18, GRW18] and related groups [CM11, RW13b, BM13, Kup15, Til16, Nar17, Gre19, Kra20].
- Various other groups: *automorphism groups of right-angled Artin groups* [GW16], *Artin monoids* [Boy20], *Houghton groups* [PW16], *Higman–Thompson groups* [SW19], *free nilpotent groups* [Szy14], and Coxeter groups [Hep16].

A somewhat outdated but still interesting survey of stability phenomena is [Coh09].

2.1. When should you expect homological stability? I have three reasons to introduce Theorem 2.2:

- (1) It abstracts those properties of symmetric groups that we use.
- (2) It allows me to state a generalisation with local coefficients in the next section.
- (3) It leads to a heuristic for when one expects a sequence of classifying spaces of groups to exhibit homological stability.

It is this last point I want to make more explicit by stating Theorem 2.2 informally. Suppose that if you have a natural way to “summing” old objects together to make new ones (a monoidal structure), which is sufficiently symmetric (this monoidal structure is symmetric or more generally braided) and somewhat reasonable (properties (i) and (ii) in Theorem 2.2). Then you should expect homological stability to hold for the automorphism groups of $A \oplus X^{\oplus n}$ exactly when the “space of ways of removing copies of X from $A \oplus X^{\oplus n}$ ” increases in connectivity with n . As this description suggests, the last property is the hardest one to verify and doing so is more of an art than a science. It requires new ideas specific to each situation.¹

3. LOCAL COEFFICIENTS AND A THEOREM OF DWYER

3.1. Homological stability with polynomial coefficients. In an exercise for the previous lecture, we considered replacing the constant coefficients \mathbb{Z} by certain local coefficients. In the general framework of Theorem 2.2, these are as follows:

Definition 3.1. Let $\mathcal{U}G_{A,X}$ be the full subcategory of $\mathcal{U}G$ on objects $A \oplus X^{\oplus n}$. Then a *coefficient system* is a functor $F: \mathcal{U}G_{A,X} \rightarrow \text{Ab}$.

More concretely, this provides a collection of $F_n := F(A \oplus X^{\oplus n})$ of abelian groups with G_n -action together with G_n -equivariant morphisms $F_n \rightarrow F_{n+1}$. In particular, we can ask whether the map

$$\sigma_*: H_*(BG_n; F_n) \longrightarrow H_*(BG_{n+1}; F_{n+1})$$

¹This is one of the reasons for including the long list of references above; if you want to prove a connectivity result for a semi-simplicial set, look at such a result in a related situation.

is an isomorphism in a range. That is, does the sequence $BG_0 \xrightarrow{\sigma} BG_1 \xrightarrow{\sigma} \cdots$ exhibit homological stability with coefficients in F ? This is the case for those coefficient system where he can reduce to trivial coefficients.

There is a functor $\Sigma_X: \mathcal{U}G_{A,X} \rightarrow \mathcal{U}G_{A,X}$ given on objects by sending $A \oplus X^{\oplus n}$ to $A \oplus X^{\oplus n+1}$ and morphism f to $(\beta_{A,X}^{-1} \oplus \text{id}) \circ (X \oplus f) \circ (\beta_{A,X} \oplus \text{id})$. There is a natural transformation $\sigma_X: \text{id} \rightarrow \Sigma_X$.

Definition 3.2.

- $\text{coker}(F) := \text{coker}(F \rightarrow \Sigma_X F)$ and $\text{ker}(F) := \text{ker}(F \rightarrow \Sigma_X F)$.
- F is *polynomial of degree -1* if $F(X \oplus A^{\oplus n}) = 0$ for sufficiently large n .
- F is *polynomial of degree r* if $\text{coker}(F)$ is polynomial of degree $r-1$ and $\text{ker}(F)$ is polynomial of degree -1 .

Example 3.3. F is polynomial of degree 0 if it is eventually constant.

By reduction to the constant coefficients, we can deduce a homological stability with coefficients in a polynomial functor of finite degree.

Theorem 3.4 ([RWW17]). *Under the assumptions of Theorem 2.2, if F is polynomial of degree r , there exists an $N \geq 0$ such when $n \geq N$, the map*

$$\sigma_*: H_*(BG_n; F_n) \longrightarrow H_*(BG_{n+1}; F_{n+1})$$

is a surjection for $ \leq \frac{n}{k} - r$ and an isomorphism for $* \leq \frac{n}{k} - r - 1$.*

3.2. Dwyer’s finiteness theorem. Let us give an application of this, due to Dwyer. Recall homological stability for symmetric groups proved that $\pi_*(\mathbb{S})$ is finite for $* > 0$. We can prove a similar result for $K(\mathbb{S})$, the algebraic K -theory spectrum of the sphere spectrum (also known as $A(*)$):

Theorem 3.5 ([Dwy80]). *$\pi_*(K(\mathbb{S}))$ is finitely generated for $* > 0$.*

By Waldhausen’s work [Wal85], the infinite loop space $\Omega^\infty K(\mathbb{S})$ can be obtained as the group completion of the topological monoid

$$\bigsqcup_{n \geq 0} \text{hocolim}_{d \rightarrow \infty} \text{BhAut}_*(\vee_n S^d),$$

with map $\text{hAut}_*(\vee_n S^d) \rightarrow \text{hAut}_*(\vee_n S^{d+1})$ induced by suspension and multiplication map induced by wedging. For the sake of brevity, we abbreviate the term $\text{hocolim}_{d \rightarrow \infty} \text{BhAut}_*(\vee_n S^d)$ to $\text{BhAut}_*(\vee_n \mathbb{S})$.

We may as well assume that $d \geq 2$. The action on H_d then induces an isomorphism $\pi_0(\text{hAut}_*(\vee_n S^d)) \cong \text{GL}_n(\mathbb{Z})$. We can further understand the higher homotopy groups with their $\text{GL}_n(\mathbb{Z})$ -action using the Freudenthal suspension theorem and Hilton–Milnor theorem: for $* \leq d$, the suspension map $\pi_*(\text{hAut}_*(\vee_n S^d)) \rightarrow \pi_*(\text{hAut}_*(\vee_n S^{d+1}))$ is an isomorphism, and as a $\mathbb{Z}[\text{GL}_n(\mathbb{Z})]$ -module we have

$$\pi_*(\text{hAut}_*(\vee_n S^d)) \cong \pi_*(\mathbb{S}) \otimes_{\mathbb{Z}} \text{Ad}_n \quad \text{for } 0 < * \leq d,$$

where Ad_n is given by $\text{GL}_n(\mathbb{Z})$ acting on $(n \times n)$ -matrices with integral entries by conjugation and the $\text{GL}_n(\mathbb{Z})$ -action on $\pi_*(\mathbb{S})$ is trivial. These descriptions are compatible with stabilisation.

As a consequence, the group $\pi_*(\text{BhAut}_*(\vee_n \mathbb{S}))$ for $* = 1$ is $\text{GL}_n(\mathbb{Z})$ and the coefficient system $\pi_*(\text{BhAut}_*(\vee_{(-)} \mathbb{S}))$ for $* > 1$ lies in the class \mathcal{P} of polynomial functors which are of finite degree and objectwise finitely-generated. This class is quite well-behaved:

Lemma 3.6.

(i) \mathcal{P} is closed under passing to subobjects, quotients, and extensions: in an exact sequence

$$0 \longrightarrow F \longrightarrow G \longrightarrow Q \longrightarrow 0$$

of coefficients systems, $F, Q \in \mathcal{P}$ if and only if $G \in \mathcal{P}$.

(ii) \mathcal{P} is closed under tensor products and Tor: if $F, G \in \mathcal{P}$ then $F \otimes_{\mathbb{Z}} G$ and $\mathrm{Tor}_{\mathbb{Z}}^1(F, G)$ are in \mathcal{P} .

(iii) \mathcal{P} is closed under taking homology of Eilenberg–Mac Lane spaces: if $F, G \in \mathcal{P}$ then $H_q(K(F, n); G) \in \mathcal{P}$ for all $q, n > 0$.

The proof of this lemma is quite involved, and will appear in a forthcoming joint paper with Manuel Krannich. Dwyer worked with a more restricted class sufficient for the purpose of proving Theorem 3.5.

We will use Lemma 3.6 to prove two qualitative statements about the homology of $\mathrm{BhAut}_*(\vee_n \mathbb{S})$. Let $\mathrm{hAut}_*^{\mathrm{id}}(\vee_n S^d) \subset \mathrm{hAut}_*(\vee_n S^d)$ be the path component containing the identity map. For all $n \geq 0$, there is then a fibration sequence

$$\mathrm{hocolim}_{d \rightarrow \infty} \mathrm{BhAut}_*^{\mathrm{id}}(\vee_n S^d) \longrightarrow \mathrm{hocolim}_{d \rightarrow \infty} \mathrm{BhAut}_*(\vee_n S^d) \longrightarrow \mathrm{BGL}_n(\mathbb{Z}),$$

compatible with stabilisation and suspension. Letting $d \rightarrow \infty$ we thus get Serre spectral sequences

$$H_p(\mathrm{BGL}_n(\mathbb{Z}); H_q(\mathrm{BhAut}_*^{\mathrm{id}}(\vee_n \mathbb{S}); \mathbb{Z})) \implies H_{p+q}(\mathrm{BhAut}_*(\vee_n \mathbb{S}); \mathbb{Z})$$

for all $n \geq 0$, connected by stabilisation maps.

A standard Serre class argument over the Postnikov tower of $\mathrm{BhAut}_*^{\mathrm{id}}(\vee_n \mathbb{S})$ using Lemma 3.6 implies the following, see Exercise 4.7:

Lemma 3.7. *For all $q \geq 0$, the coefficient system $H_q(\mathrm{BhAut}_*^{\mathrm{id}}(\vee_{(-)} \mathbb{S}); \mathbb{Z})$ lies in \mathcal{P} .*

Proposition 3.8. *The sequence $\mathrm{BhAut}_*(\vee_0 \mathbb{S}) \xrightarrow{\sigma} \mathrm{BhAut}_*(\vee_1 \mathbb{S}) \xrightarrow{\sigma} \dots$ exhibits homological stability.*

Proof. Applying spectral sequence comparison to the maps of spectral sequences

$$\begin{array}{c} H_p(\mathrm{BGL}_n(\mathbb{Z}); H_q(\mathrm{BhAut}_*^{\mathrm{id}}(\vee_n \mathbb{S}); \mathbb{Z})) \implies H_{p+q}(\mathrm{BhAut}_*(\vee_n \mathbb{S}); \mathbb{Z}) \\ \downarrow \\ H_p(\mathrm{BGL}_{n+1}(\mathbb{Z}); H_q(\mathrm{BhAut}_*^{\mathrm{id}}(\vee_{n+1} \mathbb{S}); \mathbb{Z})) \implies H_{p+q}(\mathrm{BhAut}_*(\vee_{n+1} \mathbb{S}); \mathbb{Z}), \end{array}$$

we see it suffices to prove that for all $q \geq 0$ the maps

$$\sigma_* : H_*(\mathrm{BGL}_n; H_q(\mathrm{BhAut}_*^{\mathrm{id}}(\vee_n \mathbb{S}); \mathbb{Z})) \longrightarrow H_*(\mathrm{BGL}_{n+1}; H_q(\mathrm{BhAut}_*^{\mathrm{id}}(\vee_{n+1} \mathbb{S}); \mathbb{Z}))$$

are isomorphisms in a range of degrees $*$ tending to ∞ with n . This follows by combining Theorem 2.2 with Example 2.5 and Lemma 3.7. \square

The previous argument only uses that the coefficient systems $H_q(\mathrm{BhAut}_*^{\mathrm{id}}(\vee_{(-)} \mathbb{S}); \mathbb{Z})$ are polynomial of finite degree, not that they are objectwise finitely generated. This instead is used to prove:

Proposition 3.9. *Fixing n , $H_*(\mathrm{BhAut}_*(\vee_n \mathbb{S}); \mathbb{Z})$ is finitely-generated for each $* \geq 0$.*

Proof. This uses once more the spectral sequence

$$H_p(BGL_n(\mathbb{Z}); H_q(\mathrm{BhAut}_*^{\mathrm{id}}(\vee_n \mathbb{S}); \mathbb{Z})) \implies H_{p+q}(\mathrm{BhAut}_*(\vee_n \mathbb{S}); \mathbb{Z}).$$

Recall that if X is a CW-complex with finitely many cells in each dimension, then for any local coefficient system \mathcal{A} which is finitely generated as an abelian group, the homology groups $H_*(X; \mathcal{A})$ are finitely generated in each degree. Since each group $H_q(\mathrm{BhAut}_*^{\mathrm{id}}(\vee_n \mathbb{S}); \mathbb{Z})$ is finitely generated, it suffices to prove that $BGL_n(\mathbb{Z})$ has the homotopy type of a CW-complex with finitely many cells in each dimension. This is a result of Borel–Serre [BS73, §11.1]. \square

Proof of Theorem 3.5. The space $\Omega_0^\infty K(\mathbb{S})$ is simple, so by a Serre class argument its homotopy groups are finitely generated if and only if its homology groups are. By Waldhausen’s work and the group completion theorem, the homology of $\Omega_0^\infty K(\mathbb{S})$ is equal to the stable homology of $\mathrm{BhAut}_*(\vee_n \mathbb{S})$. By Proposition 3.8 the map

$$H_*(\mathrm{BhAut}_*(\vee_n \mathbb{S}); \mathbb{Z}) \longrightarrow \operatorname{colim}_{n \rightarrow \infty} H_*(\mathrm{BhAut}_*(\vee_n \mathbb{S}); \mathbb{Z})$$

is an isomorphism in a range tending to ∞ with n . That the right side is finitely generated in all degrees thus follows from Proposition 3.9. \square

4. EXERCISES

Exercise 4.1 (Coning off).

- (i) Let X_\bullet be a semi-simplicial set and $\operatorname{simp}(X_\bullet)$ be the poset with objects given by the simplices X_\bullet and $\sigma \leq \tau$ if σ can be obtained from τ by applying face maps. Prove that $\|X_\bullet\|$ is homeomorphic to $|N(\operatorname{simp}(X_\bullet))|$. (Hint: Barycentric subdivision.)
- (ii) Identify $\operatorname{simp}(W_n(\underline{1})_\bullet)$ with the poset $I(\underline{n})$ of ordered non-empty subsets of \underline{n} , ordered by order-preserving inclusions.
- (iii) For $S_i = \underline{n} \setminus \{i\}$, construct a zigzag of natural transformations between the inclusion $I(S_i) \rightarrow I(\underline{n})$ and the constant map $I(S_i) \rightarrow I(\underline{n})$ with value (i) .
- (iv) Conclude that $\|W(S_i)_\bullet\| \rightarrow \|W(\underline{n})_\bullet\|$ is null-homotopic, as claimed in the proof of Proposition 1.1.

Exercise 4.2 (Simplicial complexes). A *simplicial complex* X is a set V (called *vertices*) and a collection S of unordered finite subsets of V (called *simplices*) satisfying (a) $\{v\} \in S$ for all $v \in V$, (b) if $\sigma \in S$ then any subset of σ is in S as well. If $\sigma \in S$ has $p+1$ elements we call it a *p -simplex*. The geometric realisation $|X|$ is defined taking

$$|X| = \left(\bigsqcup_k \Delta^k \times \{k\text{-simplices of } X\} \right) / \sim$$

where the equivalence relation \sim is similar to that for the geometric realisation of a semi-simplicial space (we leave the details for the reader).

- (i) From a simplicial complex X one can extract a semi-simplicial set $X_\bullet^{\operatorname{ord}}$ by taking its p -simplices to be ordered $(p+1)$ -element subsets of V whose underlying unordered set is a p -simplex in X . Describe the face maps and verify this is indeed a semi-simplicial set.
- (ii) For a finite set V , let Δ^V be the simplicial complex where each finite subset of V is a simplex. Prove that $|\Delta^V|$ is homeomorphic to $\Delta^{\#V-1}$ and $\operatorname{ord}_\bullet(\Delta^V) = W_{\#V}(\underline{1})_\bullet$.

- (iv) A *link* of a simplex σ of a simplicial complex X consists of all simplices τ such that $\sigma \cap \tau = \emptyset$ and $\sigma \cup \tau$ is a simplex. Describe how this can be made into a simplicial complex $\text{link}_X(\sigma)$ and show that $\text{link}_{\Delta^V}(Y)$ for a subset $Y \subset V$ is isomorphic to $\Delta^{V \setminus W}$.
- (v) A simplicial complex X is said to be *weakly Cohen–Macaulay of dimension d* if (a) $|X|$ is $(d - 1)$ -connected and (b) for all k -simplices σ and $k \geq 0$, $|\text{link}_X(\sigma)|$ is $(d - k - 2)$ -connected. Show that Δ^V is weakly Cohen–Macaulay of dimension $\#V - 1$.
- (vi) By [RWW17, Proposition 2.14], if X is weakly Cohen–Macaulay of dimension d then $\|X_{\bullet}^{\text{ord}}\|$ is $(d - 1)$ -connected. Use this to prove that $\|W_n(\mathbb{1})_{\bullet}\|$ is $(n - 2)$ -connected.

Exercise 4.3 (An example). Prove that $\|W_2(\mathbb{Z}, \mathbb{Z})_{\bullet}\|$ from Example 2.5 is path-connected.

Exercise 4.4 (Homological stability for hyperoctahedral groups). Let \mathbf{G} be the symmetric monoidal groupoid of finite free $\mathbb{Z}/2$ -sets, with monoidal structure given by disjoint union.

- (i) For $X = \mathbb{Z}/2$, prove that $\text{Aut}(X^{\oplus n}) = \mathbb{Z}/2 \wr \Sigma_n$, the n th *hyperoctahedral group*.
- (ii) Describe $W_n(\emptyset, X)_{\bullet}$ and prove that it is homologically $\frac{n-1}{2}$ -connected along the lines of Proposition 1.1.
- (iii) Deduce a homological stability result for hyperoctahedral groups.

Exercise 4.5 (A non-injective stabilisation map). Theorem 2.2 also generalises to give homological stability with abelian coefficients. You may assume this (with an unspecified range) in this exercise.

- (i) Use this to prove that the sequence $B\text{SL}_0(\mathbb{Z}) \xrightarrow{\sigma} B\text{SL}_1(\mathbb{Z}) \xrightarrow{\sigma} \dots$ exhibits homological stability.
- (ii) Use the facts that (a) $\pm \text{id} \subset \text{SL}_2(\mathbb{Z})$ is the center and (b) $\text{SL}_2(\mathbb{Z})/\{\pm \text{id}\} \cong \mathbb{Z}/2 * \mathbb{Z}/3$, to prove that $H_1(B\text{SL}_2(\mathbb{Z}); \mathbb{Z}) \cong \mathbb{Z}/12$.
- (iii) Use the fact that $\text{SL}_n(\mathbb{Z})$ is perfect for $n \geq 3$ to prove that the stabilisation map $\sigma_*: H_1(B\text{SL}_2(\mathbb{Z}); \mathbb{Z}) \rightarrow H_1(B\text{SL}_3(\mathbb{Z}); \mathbb{Z})$ is not injective.

Exercise 4.6 (Algebraic K -theory of the integers). Instead of asking about the algebraic K -theory of \mathbb{S} , one could ask about that of \mathbb{Z} . These are the homotopy groups of a spectrum $K(\mathbb{Z})$ so that $\Omega^\infty K(\mathbb{Z})$ is the group completion of the topological monoid $\bigsqcup_{n \geq 0} B\text{GL}_n(\mathbb{Z})$.

- (i) Explain that $\pi_*(K(\mathbb{Z}))$ is finitely generated for $* > 0$ if and only if $H_*(\Omega_0^\infty K(\mathbb{Z}); \mathbb{Z})$ is finitely generated for $* > 0$.
- (ii) Explain that there is an isomorphism $H_*(\Omega_0^\infty K(\mathbb{Z}); \mathbb{Z}) \cong \text{colim}_{n \rightarrow \infty} H_*(B\text{GL}_n(\mathbb{Z}); \mathbb{Z})$.
- (iii) Prove that right term in (ii) is finitely generated for all $* > 0$ using Theorem 2.2, Example 2.5, and the result of Borel–Serre.

Exercise 4.7 (Some polynomial coefficients systems). Use Lemma 3.6 to prove Lemma 3.7.

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