Exotic $K(h)$-local Picard groups when $2p-1=h^2$ and the Vanishing Conjecture

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1. $K(h)$-local Picard groups

2. Duality

3. Greek letter elements

Overview

Theorem (Hopkins-Mahowald-Sadofsky)

The exotic $K(h)$-local Picard group $\kappa_h = 0$ at primes p such that $(p-1) \nmid h$ and $2p - 1 > h^2$.

In this joint work in progress with Dominic Culver, we study κ_h at prime p such that $2p - 1 = h^2$.

Observation

- The assumption $2p - 1 = h^2$ implies $(p-1) \nmid h$.
- Unknown if there are infinitely many such pairs of primes and heights.

Main results

Theorem (Culver-Z.)

Suppose the prime p and the height h satisfy $2p-1=h^2$.

1. *When the Smith-Toda complex $V(h-2)$ exists, elements X in κ_h cannot be detected by $V(h-2)$, i.e. $X \otimes_{K(h)} V(h-2) \simeq L_{K(h)} V(h-2)$. (e.g. $(h,p)=(3,5),(5,13)$)*
2. *If the Reduced Homological Vanishing Conjecture holds at $p=5$ and $h=3$, then $\kappa_3=0$ at $p=5$.*
3. *There are bounds on the divisibility of Greek letter elements that would imply both RHVC and $\kappa_h=0$ when $2p-1=h^2$.*

Conjecture (Reduced Homological Vanishing Conjecture)

$$\mathbb{F}_p \simeq H_0(\mathbb{G}_h;\mathbb{F}_{p^h}) \xrightarrow{\sim} H_0(\mathbb{G}_h;\pi_0(E_h)/p)$$

Strategy

1. GHMR defined a map $\tau : \kappa_h \to H_c^{2p-1}(\mathbb{G}_h; \pi_{2p-2}(E_h))$, which is injective when $2p-1 = h^2$.
2. Show that $H_c^{h^2}(\mathbb{G}_h; M) \simeq H_c^{h^2}(\mathbb{G}_h; M/p)$ for $M = \pi_{2p-2}(E_h)$.
3. Use Poincaré duality to relate $H_c^{h^2}$ to both H_0 and H_c^0:

$$H_c^{h^2}(\mathbb{S}_h; M) \simeq H_0(\mathbb{S}_h; M) \qquad H_c^{h^2}(\mathbb{S}_h; M) \simeq H_c^0(\mathbb{S}_h; M^\vee)^\vee$$

4. The Gross-Hopkins duality identifies the equivariant Pontryagin dual for $\pi_t(E_h)$ as an $\mathbb{S}_h$-E_h-module. This dual involves $E_h\langle\det\rangle$.
5. Identify $E_h\langle\det\rangle/p$ as a limit of finite suspensions.
6. Use the change of rings theorem to translate to BP computations.
7. Greek letter computations.
8. Use the same method to study RHVC and compare.

$K(h)$-local Picard groups

The $K(h)$-local Picard group

Definition

Denote by $\mathrm{Pic}_{K(h)}$ the Picard group of the symmetric monoidal category $(\mathrm{Sp}_{K(h)}, \otimes_{K(h)}, \mathbb{1}_{K(h)})$ where

$$X \otimes_{K(h)} Y := L_{K(h)}(X \wedge Y), \mathbb{1}_{K(h)} := S^0_{K(h)}.$$

Theorem (Hopkins-Mahowald-Sadofsky)

The followings are equivalent:

- $X \in \mathrm{Pic}_{K(h)}$.
- $(E_h)_* X$ *is a graded invertible* $(E_h)_*$*-module.*

From there, we get the zeroth detection map:

$$\begin{aligned} \mathrm{ev}_0 : \mathrm{Pic}_{K(h)} &\longrightarrow \mathrm{Pic}(\text{graded } (E_h)_*\text{-modules}) \simeq \mathbb{Z}/2 \\ X &\longmapsto (E_h)_*(X) \end{aligned}$$

The algebraic $K(h)$-local Picard group

Let $\mathsf{Pic}^0_{K(h)} = \{X \in \mathsf{Pic}_{K(h)} \mid (E_h)_* X \simeq (E_h)_*\} = \ker \mathsf{ev}_0$. For any $X \in \mathsf{Pic}^0_{K(h)}$, $(E_h)_0(X)$ comes with a $\mathbb{G}_h$-action. This induces the first detection map:

$$\begin{aligned} \mathsf{ev}_1 : \mathsf{Pic}^0_{K(h)} &\longrightarrow \mathsf{Pic}(\mathbb{G}_h\text{-}(E_h)_0\text{-modules}) := \mathsf{Pic}^{alg,0}_{K(h)} \\ X &\longmapsto (E_h)_0(X) \end{aligned}$$

Theorem (Hopkins-Mahowald-Sadofsky)

$\mathsf{Pic}^{alg,0}_{K(h)} \simeq H^1_c(\mathbb{G}_h; \pi_0(E_h)^\times)$.

Example

When $h = 1$, $\mathsf{Pic}^{alg,0}_{K(1)} \simeq \mathrm{End}_c(\mathbb{Z}_p^\times) \simeq \begin{cases} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}_2 & p = 2; \\ \mathbb{Z}/(p-1) \oplus \mathbb{Z}_p & p > 2. \end{cases}$

The exotic $K(h)$-local Picard group

Definition

$\kappa_h := \ker \mathrm{ev}_1$ is called the exotic $K(h)$-local Picard group.

Homotopy groups of $X \in \mathsf{Sp}_{K(h)}$ are computed by the HFPSS:

$$E_2^{s,t}(X) = H_c^s(\mathbb{G}_h; (E_h)_t(X)) = H_c^s(\mathbb{S}_h; (E_h)_t(X))^{\mathsf{Gal}} \Longrightarrow \pi_{t-s}(X)$$

The E_2-page of the HFPSS for $X \in \kappa_h$ is the same as that for $S^0_{K(h)}$. Their differences lie in the higher differentials.

Theorem (Hopkins-Mahowald-Sadofsky)

$\kappa_h = 0$ when $(p-1) \nmid h$ and $2p-1 > h^2$.

Question

Is $\kappa_h = 0$ when $2p-1 = h^2$?

Proof of the Theorem

Let $X \in \kappa_h$ be an exotic element. The proof consists of four steps:

1. (0-th line) $E_2^{0,0}(X) = \mathbb{Z}_p$ and $E_2^{0,t}(X) = 0$ when $t \neq 0$.
2. (Sparseness) $E_2^{s,t}(X) = 0$ unless $2(p-1) \mid t$. This implies the first possible non-trivial differential is $d_{2p-1}^X : E_2^{0,0}(X) \to E_2^{2p-1,2p-2}(X)$.
3. (Horizontal vanishing line) When $(p-1) \nmid h$, $\mathrm{cd}_p(\mathbb{S}_h) = h^2$. This implies $E_2^{s,t}(X) = 0$ when $s > h^2$.
4. The above implies that there is no room for higher differentials in the HFPSS for X when $(p-1) \nmid h$ and $2p-1 > h^2$. As a result, any generator $[\eta] \in E_2^{0,0}(X) = \mathbb{Z}_p$ is a permanent cycle, which converges to some element $\eta \in \pi_0(X)$. One check that η factorizes as

$$\eta : S^0 \xrightarrow{L_{K(h)}} S^0_{K(h)} \xrightarrow{\sim} X$$

Algebraic detection of κ_h

GHMR used d_{2p-1} to define an algebraic detection map for κ_h:

$$\mathrm{ev}_2 : \kappa_h \longrightarrow H_c^{2p-1}(\mathbb{G}_h; \pi_{2p-2}(E_h))$$

Construction (Goerss-Henn-Mahowald-Rezk)

Fix an $\mathbb{G}_h$-equivariant isomorphism $f^X : (E_h)_* \xrightarrow{\sim} (E_h)_*(X)$ and define ϕ^X via the following commutative diagram:

$$\begin{tikzcd}
H_c^0(\mathbb{G}_h; \pi_0(E_h)) \arrow[r, dashed, "\phi^X"] \arrow[d, "(f^X)_*"', "\simeq"] & H_c^{2p-1}(\mathbb{G}_h; \pi_{2p-2}(E_h)) \arrow[d, "\simeq"', "(f^X)_*"] \\
H_c^0(\mathbb{G}_h; (E_h)_0(X)) \arrow[r, "d_{2p-1}^X"] & H_c^{2p-1}(\mathbb{G}_h; (E_h)_{2p-2}(X))
\end{tikzcd}$$

Set $\mathrm{ev}_2(X) \coloneqq \phi^X(1)$. Then ev_2 is a well-defined group homomorphism.

An exhaustive filtration on κ_h

By considering higher and higher differentials in HFPSS, we get an exhaustive filtration on κ_h, where $\kappa_h^{(m)} = \ker \mathrm{ev}_{m+1}$.

$$
\begin{array}{ccc}
\cdots & & \cdots \\
\cup\!\!| & & \\
\kappa_h^{(m)} & \xrightarrow{\mathrm{ev}_{m+2}} & E_{2m(p-1)+2}^{2(m+1)(p-1)+1,2(m+1)(p-1)} \\
\cup\!\!| & & \\
\cdots & & \cdots \\
\cup\!\!| & & \\
\kappa_h^{(1)} & \xrightarrow{\mathrm{ev}_3} & E_{2p}^{4p-3,4p-4} \\
\cup\!\!| & & \\
\kappa_h & \xrightarrow{\mathrm{ev}_2} & E_2^{2p-1,2p-2} = H_c^{2p-1}(\mathbb{G}_h; \pi_{2p-2}(E_h))
\end{array}
$$

Corollary

ev_2 *is injective when* $(p-1) \nmid h$ *and* $4p-3 > h^2$. *As a result,* $H_c^{2p-1}(\mathbb{G}_h; \pi_{2p-2}(E_h)) = 0$ *implies* $\kappa_h = 0$ *under the same assumption.*

The Chromatic Vanishing Conjecture

Conjecture (Hopkins)

The inclusion $\iota : \mathbb{W}\mathbb{F}_{p^h} \hookrightarrow \pi_0(E_h)$ of $\mathbb{G}_h$-modules induces an isomorphism on group (co)homology in all heights, primes, and degrees:

$$\iota_* : H_c^s(\mathbb{G}_h; \mathbb{W}\mathbb{F}_{p^h}) \xrightarrow{\sim} H_c^s(\mathbb{G}_h; \pi_0(E_h))$$
$$\iota_* : H_s(\mathbb{G}_h; \mathbb{W}\mathbb{F}_{p^h}) \xrightarrow{\sim} H_s(\mathbb{G}_h; \pi_0(E_h))$$

The cohomological version of the conjecture has been proved in the following cases:

- $s = 0$ for all h and p.
- $h \le 2$ for all p and s. (SY95, Beh12, Koh13, GHM14, BGH17, BDM+18, ...)

Conjecture (Reduced Homological Vanishing Conjecture)

$$\mathbb{F}_p \simeq H_0(\mathbb{G}_h; \mathbb{F}_{p^h}) \xrightarrow{\sim} H_0(\mathbb{G}_h; \pi_0(E_h)/p)$$

$\text{Pic}_{K(h)}$ for $h = 1, 2$

- (HMS) When $h = 1$, $\text{Pic}_{K(1)} = \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}/4 \times \mathbb{Z}/2 & p = 2; \\ \mathbb{Z}_p \times \mathbb{Z}/2(p-1) & p > 2. \end{cases}$
  - $\text{ev}_1 : \text{Pic}^0_{K(1)} \to \text{Pic}^{alg,0}_{K(1)}$ is surjective.
  - HMS's Theorem implies $\kappa_1 = 0$ when $p > 2$.
  - When $p = 2$, $\text{ev}_2 : \kappa_1 \xrightarrow{\sim} H^3_c(\mathbb{G}_1; \pi_2(E_1)) = \mathbb{Z}/2$.
- When $h = 2$ and $p \geq 3$, $\text{Pic}_{K(2)} = \begin{cases} \mathbb{Z}_3^2 \times \mathbb{Z}/16 \times \mathbb{Z}/3 \times \mathbb{Z}/3 & p = 3; \\ \mathbb{Z}_p^2 \times \mathbb{Z}/2(p^2-1) & p \geq 5. \end{cases}$
  - $\text{ev}_2 : \text{Pic}^0_{K(2)} \to \text{Pic}^{alg,0}_{K(2)}$ is surjective.
  - (Hopkins) When $p \geq 5$, $\text{Pic}^{alg,0}_{K(2)} = \mathbb{Z}_p^2 \times \mathbb{Z}/(p^2-1)$. This computation uses the Vanishing Conjecture at $s = 1$.
  - (Karamanov) The above formula holds at $p = 3$.
  - HMS's Theorem implies $\kappa_2 = 0$ when $p \geq 5$.
  - (GHMR) When $p = 3$, $\text{ev}_2 : \kappa_2 \xrightarrow{\sim} H^5_c(\mathbb{G}_2; \pi_4(E_2)) = \mathbb{Z}/3 \times \mathbb{Z}/3$.

Duality

Bounded torsion

Proposition

Let $M = \pi_{2p-2}(E_h)$. *Then* $H_c^{h^2}(\mathbb{G}_h; M) \simeq H_c^{h^2}(\mathbb{G}_h; M/p)$ *if* $(p-1) \nmid h$.

Proof.

From the SES of $\mathbb{G}_h$-presentations

$$0 \longrightarrow M \xrightarrow{-\cdot p} M \longrightarrow M/p \longrightarrow 0 \ ,$$

we get a LES of group cohomology of $\mathbb{G}_h$. By considering the action of the center $\mathbb{Z}_p^\times \trianglelefteq \mathbb{S}_h$ on M, we get a Lyndon-Hochschild-Serre SS:

$$E_2^{r,s} = H_c^r(\mathbb{G}_h/\mathbb{Z}_p^\times; H_c^s(\mathbb{Z}_p^\times; M)) \Longrightarrow H_c^{r+s}(\mathbb{G}_h; M).$$

This SS implies $H_c^*(\mathbb{G}_h; M)$ is p-torsion. Now from the LES in group cohomology, we conclude $H_c^{h^2}(\mathbb{G}_h; M) \simeq H_c^{h^2}(\mathbb{G}_h; M/p)$, since $h^2 = \mathrm{cd}_p(\mathbb{S}_h)$ when $(p-1) \nmid h$. □

Poincaré duality

Goal

Compute $H_c^{h^2}(\mathbb{S}_h; \pi_t(E_h)/p)^{\mathrm{Gal}}$ when $(p-1) \nmid h$ and $t = 0$ or $2p-2$.

A direct computation of $H_c^{h^2}$ when $h \geq 3$ seems to be out of reach. Instead, we use Poincaré duality to relate this computation to H_0 and H_c^0.

Theorem (Lazard, Symonds-Weigel)

$\mathbb{S}_h$ is a p-adic Poincaré duality group of dimension h^2.More precisely, let M be a p-complete $\mathbb{S}_h$-representation. Then we have

$$H_c^s(\mathbb{S}_h; M) \simeq H_{h^2-s}(\mathbb{S}_h; M) \qquad H_c^s(\mathbb{S}_h; M) \simeq H_c^{h^2-s}(\mathbb{S}_h; M^\vee)^\vee,$$

where $(-)^\vee := \mathrm{Hom}(-, \mathbb{Q}_p/\mathbb{Z}_p)$ is the p-adic Pontryagin dual.

Corollary

RHVC is equivalent to the mod-p CVC at $s = h^2$ when $(p-1) \nmid h$.

Gross-Hopkins duality

We will use the second form of Poincaré duality, and relate it to the Greek letter computations. To do so, we must identity the $\mathbb{S}_h$-equivariant Pontryagin dual $M^\vee$ for $M = \pi_t(E_h)/p$.

Theorem (Gross-Hopkins)

Let $\mathfrak{m} = (p, u_1, \cdots, u_{h-1}) \trianglelefteq \pi_0(E_h)$ *be the maximal ideal, then*

$$\pi_t(E_h)^\vee \simeq \pi_{2h-t} E_h\langle\det\rangle/\mathfrak{m}^\infty,$$

where $\langle\det\rangle$ *denotes the determinant twist of the stabilizer group action.*

The determinant twist mod p

So far, we have showed:

$$H_c^{h^2}(\mathbb{S}_h; \pi_t(E_h)/p) \simeq H_c^0(\mathbb{S}_h; \pi_{2h-t}(E_h)\langle\det\rangle/(p)+\mathfrak{m}^\infty)^\vee.$$

The next step is to identify the determinant twist mod p.

Theorem (Gross-Hopkins)

There is an isomorphism of $\mathbb{G}_h$-modules:

$$\pi_*(E_h)\langle\det\rangle/p \simeq \pi_*\left(\Sigma^{\lim\limits_{N\to\infty}\frac{p^N|v_h|}{p-1}} E_h\right)\Big/ p.$$

More precisely, let $J \trianglelefteq \pi_0(E_h)$ be an open invariant ideal containing p, such that $(E_h)_*/J$ has a $v_h^{p^N}$-self map, then

$$\pi_t(E_h)\langle\det\rangle/J \simeq \pi_t\left(\Sigma^{\frac{p^N|v_h|}{p-1}} E_h\right)\Big/ J = \pi_{t-\frac{p^N|v_h|}{p-1}}(E_h)/J.$$

Fixed points of the quotient mod $(p) + \mathfrak{m}^\infty$

Let M be a $\mathbb{G}_h$-$\pi_0(E_h)$-module. Recall $M/(p) + \mathfrak{m}^\infty := \operatorname*{colim}_{p \in J \trianglelefteq E_h} M/J$, where J ranges over all open invariant ideals of E_h containing p. Notice this colimit is filtered and $\mathbb{S}_h$ is topologically finitely generated. This implies the canonical map

$$\operatorname*{colim}_{p \in J \trianglelefteq E_h} H_c^0(\mathbb{S}_h; M/J) \xrightarrow{\sim} H_c^0(\mathbb{S}_h; M/(p) + \mathfrak{m}^\infty)$$

is an isomorphism. Consequently, we have established in this section:

$$H_c^{h^2}(\mathbb{G}_h; \pi_t(E_h)/p) \simeq \operatorname*{colim}_{p \in J \trianglelefteq E_h} H_c^0\left(\mathbb{S}_h; \pi_{2h-t-\frac{p^{N(J)}|v_h|}{p-1}}(E_h)/J\right)^{\mathsf{Gal}},$$

where $N(J)$ is the smallest N so that $v_h^{p^N}$ is invariant mod J. To prove LHS is 0 $(\mathbb{F}_p)$ when $t = 2p-2$ $(t = 0)$, we want to show that every single term in the colimit on RHS is zero $(\mathbb{F}_p)$.

Greek letter elements

The Change of Rings Theorem

Let M be a BP_*BP-comodule. Denote $\mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, M)$ by $H^{s,t}(M)$.

Theorem (Change of Rings)

Let $\mathcal{I}_h = (p, v_1, v_2, \cdots, v_{h-1}) \trianglelefteq BP_*$ be an invariant prime ideal and $\mathfrak{m} \trianglelefteq \pi_0(E_h)$ be the maximal ideal. Then there is an isomorphism:

$$H_c^s(\mathbb{G}_h; \pi_t(E_h)/\mathfrak{m}^\infty) \simeq H^{s,t}(v_h^{-1}BP_*/\mathcal{I}_h^\infty)$$

Let J be an invariant ideal of BP_* such that $p \in J \subseteq \mathcal{I}_h$ and that N is the smallest number so that $v_h^{p^N}$ is invariant mod J. Using the Change of Rings Theorem, we can translate our computational goals to:

$$H^{0,2h-t-\frac{p^N|v_h|}{p-1}}(v_h^{-1}BP_*/J) = \begin{cases} \mathbb{F}_p, & t = 0; \\ 0, & t = 2p-2 \end{cases} \quad \text{holds for all such } J.$$

Greek letter elements mod $\mathcal{I}_{h-1}$

Let $M^k_{h-k} = v_h^{-1}BP_*/(p, v_1, \cdots, v_{k-1}, v_k^\infty, \cdots, v_{h-1}^\infty)$.

Theorem (Miller-Ravenel-Wilson)

$H^{0,*}(M^1_{h-1})$ is additively generated by the following elements:

I. $\frac{v_h^s}{pv_1 \cdots v_{h-2}v_{h-1}}$, *where* $(p, s) = 1$.

II. $\frac{1}{pv_1 \cdots v_{h-2}v_{h-1}^j}$, *where* $j \geq 1$.

III. $\frac{x_{h,N}^s}{pv_1 \cdots v_{h-2}v_{h-1}^j}$, *where* $N \geq 1$, $x_{h,N}$ *is some replacement of* $v_h^{p^N}$, $(p, s) = 1$, *and* $1 \leq j \leq a_{h,N}$ *for some* $a_{h,N}$ *defined by a recursive formula.*

For any open invariant ideal $\mathcal{I}_{h-1} \subseteq J \subseteq \mathcal{I}_h$, $H^{0,*}(v_h^{-1}BP_*/J)$ is then generated by the elements above that satisfy $J \subseteq (p, v_1, \cdots, v_{h-2}, v_{h-1}^j)$.

$H_c^{h^2}$ mod $(p, u_1, \cdots, u_{h-2})$

Lemma

$\alpha \in H^{0,*}(M_{h-1}^1)$ corresponds some element in $H^{h^2}(\mathbb{G}_h; \pi_t(E_h)/(p, u_1, \cdots, u_{h-2}))$ iff

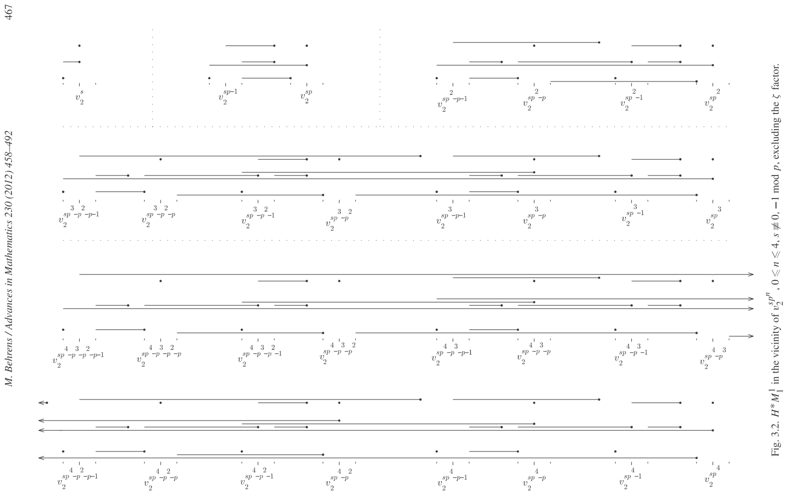
$$|\alpha| \equiv 2h - t - \frac{p^N |v_h|}{p-1} \mod p^N |v_h|$$

Corollary

Let $t \in \mathbb{Z}$. $H_c^{h^2}(\mathbb{G}_h, \pi_t(E_h)/(p, u_1, \cdots, u_{h-2}))$ is zero unless:

- *(Family I) $|v_h|$ divides t.*
- *(Family III) $t = sp^N|v_h| - \frac{(p^N-1)|v_h|}{p-1} + (j-1)|v_{h-1}|$, for some $N \geq 1$ and $1 \leq j \leq a_{h,N}$.*

When $h = 2$, $p \geq 5$, we recover Behrens' computation of $H_c^4(\mathbb{G}_2; \pi_t(E_2)/p)$.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M. Behrens / Advances in Mathematics 230 (2012) 458–492
Fig. 3.2. $H^*M_1^1$ in the vicinity of $v_2^{sp^n}$, $0 \leqslant n \leqslant 4$, $s \not\equiv 0, -1 \bmod p$, excluding the ζ factor.

Implications of the MRW computations

When $(p-1) \nmid h$, the above computation yields:

Corollary

- $H_0(\mathbb{G}_h; \pi_0(E_h)/\mathcal{I}_{h-1}) = H_c^{h^2}(\mathbb{G}_h; \pi_0(E_h)/\mathcal{I}_{h-1}) = \mathbb{F}_p$.
- $H_0(\mathbb{G}_h; \pi_{2p-2}(E_h)/\mathcal{I}_{h-1}) = H_c^{h^2}(\mathbb{G}_h; \pi_{2p-2}(E_h)/\mathcal{I}_{h-1}) = 0$.

When $2p-1 = h^2$, the latter measures whether the exotic elements in κ_h are detected by the Smith-Toda complex $V(h-2) := S^0/(p, v_1, \cdots, v_{h-2})$. From this we conclude

Theorem (Culver-Z.)

Suppose the Smith-Toda complex $V(h-2)$ exists and $2p-1 = h^2$. Then

$$X \otimes_{K(h)} V(h-2) \simeq L_{K(h)} V(h-2), \text{ for any } X \in \kappa_h.$$

In particular, this is true when $(h,p) = (3,5)$ and $(5,13)$.

Greek letter elements mod p

Proposition

As an $\mathbb{F}_p$-vector space, $H^{0,*}(M_1^{h-1})$ is generated by elements in the following families:

I. $\frac{v_h^s}{pv_1 \cdots v_{h-1}}$, *where* $(s,p) = 1$.

II. $\frac{1}{pv_1^{d_1} \cdots v_{h-1}^{d_{h-1}}}$, *where* $(p, v_1^{d_1}, \cdots, v_{h-1}^{d_{h-1}})$ *is an invariant ideal.*

III. $\frac{y_{N,h}^s}{pv_1^{d_1} \cdots v_{h-1}^{d_{h-1}}}$, *where* $y_{N,h}$ *is some replacement of* $v_h^{p^N}$, $(s,p) = 1$ *and* $(p, v_1^{d_1}, \cdots, v_{h-1}^{d_{h-1}}, y_{N,h}^s)$ *is an "admissible" sequence.*

By analyzing the degrees of elements in Families I and II, we get

- Family I contributes a copy of $\mathbb{F}_p$ to $H_c^{h^2}(\mathbb{G}_h; \pi_t(E_h)/p)$ when $|v_h| \mid t$.
- Family II does not contribute to $H_c^{h^2}(\mathbb{G}_h; \pi_t(E_h)/p)$ when $t \in \mathbb{Z}$.

Bounds on divisibility

It now remains to study Family III. The precise condition for $(p, v_1^{d_1}, \cdots, v_{h-1}^{d_{h-1}}, y_{N,h}^s)$ to be an "admissible" sequence is out of reach when $h \geq 4$. Nevertheless, we have:

Theorem (Culver-Z.)

1. *Family III does not contribute to* $H_c^{h^2}(\mathbb{G}_h; \pi_0(E_h)/p)$ *if*

$$\sum_{i=1}^{h-1} d_i |v_i| < \frac{p^N |v_h|}{p-1} - 2h \text{ for all admissible sequences.}$$

2. *Family III does not contribute to* $H_c^{h^2}(\mathbb{G}_h; \pi_{2p-2}(E_h)/p)$ *if*

$$\sum_{i=1}^{h-1} d_i |v_i| < \frac{p^N |v_h|}{p-1} - 2h + 2p - 2 \text{ for all admissible sequences.}$$

From the Vanishing Conjecture to $\kappa_h = 0$

The bounds above suggest RHVC is closely related to the vanishing of κ_h, when $2p - 1 = h^2$. Indeed we have

Theorem (Culver-Z.)

If RHVC holds at height 3 and prime 5, then $\kappa_3 = 0$ at $p = 5$.

Proof.

1. Suppose $\kappa_3 \neq 0$ at $p = 5$.
2. Then $H_c^9(\mathbb{G}_3; \pi_8(E_3)/5) \neq 0$. There is some non-zero Family III element $[\alpha] \in H_c^0(\mathbb{G}_3; (\pi_8(E_3)/5)^\vee)$.
3. $[\alpha]$ cannot be v_1-torsion, since $H_c^9(\mathbb{G}_3; \pi_8(E_3)/(5, u_1)) = 0$.
4. $v_1 \cdot [\alpha]$ corresponds to a non-zero element in $H_c^9(\mathbb{G}_3; \pi_0(E_3)/5)$.
5. $H_c^9(\mathbb{G}_3; \pi_0(E_3)/5)$ already has a copy of $\mathbb{F}_p$ from Family I elements. So $v_1 \cdot [\alpha]$ contributes another copy to it, which would imply RHVC fails.

□

Thank you!