# THE COMPUTATION OF STABLE HOMOTOPY GROUPS: PROBLEMS

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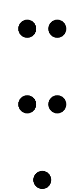
## 1. First lecture

(1) Let $A$ be an abelian group with a filtration whose associated graded object is a (graded) $\mathbb{F}_2$-vector space of dimension $n$. Prove that $A$ has order $2^n$.

(2) Let $A$ be an abelian group with a filtration whose associated graded object is a (graded) $\mathbb{F}_2$-vector space. Suppose that the associated graded object is zero in filtrations below 1 and above $n$. Prove that $2^n \cdot A$ is zero.

(3) Let $A$ be an abelian group with a filtration whose associated graded object is a (graded) $\mathbb{F}_2$-vector space. Suppose that the dimensions of $\mathrm{Gr}_0 A$, $\mathrm{Gr}_1 A$, and $\mathrm{Gr}_2 A$ are 1, 1, and 2 respectively ($\mathrm{Gr} A$ is zero in all other degrees). Find all possible values of $A$. (See the diagram below.)



(4) The diagram below represents the map $\mathrm{Gr} f : \mathrm{Gr} A \to \mathrm{Gr} B[1]$ induced by a map $f : A \to B$ of abelian groups. The solid line indicates a non-zero value of $\mathrm{Gr} f$, while the dashed line indicates a hidden value.

Analyze the map $f$, as in the example in the middle of page 3 of the lecture notes. Give names to elements of $A$ and $B$, and determine the values of $f$ in terms of these names.

[Note: This is an example of a "crossing" value. It complicates analyses of associated graded objects.]

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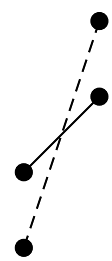

(5) The figure below represents the associated graded object of an abelian group $A$. Each dot represents a copy of $\mathbb{F}_2$ in $\mathrm{Gr}A$, and vertical lines indicate the effect of multiplication by 2 as a map $\mathrm{Gr}A \to \mathrm{Gr}A[1]$. What are the possible values of $A$?

[Note: This problem is not just interesting in principle. It actually arises in the 45-stem!]

(6) What changes if the two components in the previous diagram are separated by more than one filtration degree?

## 2. Second Lecture

(7) Check that the expression $a_{01}a_{14} + a_{02}a_{24} + a_{03}a_{34}$ in the definition of a fourfold Massey product is a cycle.

(8) Prove that

$$a_{01}\langle a_{12}, a_{23}, a_{34}\rangle = \langle a_{01}, a_{12}, a_{23}\rangle a_{34}$$

when both brackets are defined.

[Hint: First show that the indeterminacies are the same. Then show that they contain a common element.]

(9) The $E_1$-page of the May spectral sequence that converges to $\mathrm{Ext}_{A(2)}(\mathbb{F}_2, \mathbb{F}_2)$ is

$$\mathbb{F}_2[h_{01}, h_{12}, h_{23}, h_{02}, h_{13}, h_{03}],$$

with differentials

$$\begin{aligned} dh_{02} &= h_{01}h_{12} \\ dh_{13} &= h_{12}h_{23} \\ dh_{03} &= h_{01}h_{13} + h_{02}h_{23}. \end{aligned}$$

The degrees of the generators are $(0,1)$, $(1,1)$, $(3,1)$, $(2,1)$, $(5,1)$, and $(6,1)$ respectively. Compute the Massey product $\langle h_{01}, h_{12}, h_{23}, h_{12}\rangle$.

[Note: The resulting non-zero element lies in the May $E_2$-page, and it is sometimes called $h_0(1)$.]

(10) Make precise the idea that the differential graded algebra of problem (9) is the "universal threefold Massey product that contains zero".

[Hint 1: Characterize maps $E_1 \to B$ in terms of the Massey product structure on $B$.]

[Hint 2: Not all differential graded algebras are commutative, but the May $E_1$-page is commutative. This limitation has to be accounted for.]

(11) Show that the homology of the differential graded algebra from problem (9) is

$$\frac{\mathbb{F}_2[h_{01}, h_{12}, h_{23}, b_{02}, b_{13}, b_{03}, h_0(1)]}{h_{01}h_{12}, h_{12}h_{23}, h_{23}b_{02} + h_{01}h_0(1), h_{23}h_0(1) + h_{01}b_{13}, h_0(1)^2 = b_{02}b_{13} + h_{12}^2 b_{03}},$$

where $b_{ij} = h_{ij}^2$ and $h_0(1) = h_{02}h_{13} + h_{12}h_{03}$.

[Note: This is a hard, or at least lengthy, problem. But if you carry it out, then you are well on your way to computing the homotopy of tmf.]

(12) Using the Moss Convergence Theorem and Adams differentials, find Toda bracket decompositions for the following elements. [Hint: You don't need to worry about the technical conditions of the Moss Convergence Theorem. They don't pertain in these situations.]

(a) $\eta\bar{\kappa}$ detected by $h_2 f_0 = h_1 g$ in the 21-stem.

(b) $\nu\kappa$ detected by $h_0 e_0 = h_2 d_0$ in the 17-stem.

(c) A homotopy element detected by $h_2^2 h_5$.

[Note: There are two entirely different solutions to (c).]

## 3. Third lecture

(13) Let $v(n)$ be the 2-adic valuation of $n$, i.e., $2^{v(n)}$ is the highest power of 2 that divides $n$. Show that

$$v(3^k - 1) = \begin{cases} 1 & \text{if} \quad v(k) = 0 \\ 2 + v(k) & \text{if} \quad v(k) > 0. \end{cases}$$

The goal of this series of problems is to compute the cohomology of $A(1)$, which is the subalgebra of the Steenrod algebra that is generated by $\mathrm{Sq}^1$ and $\mathrm{Sq}^2$. Equivalently, this is a computation of the May spectral sequence (and Adams spectral sequence) for the homotopy of the real connective $K$-theory spectrum $ko$.

The May $E_1$-page consists of $\mathbb{F}_2[h_{01}, h_{12}, h_{02}]$, where $h_{01}$, $h_{12}$, and $h_{02}$ have degrees $(0,1)$, $(1,1)$, and $(2,1)$ respectively. Moreover, $h_{01}$ and $h_{12}$ have May filtration 0, and $h_{02}$ has May filtration 1. The May $d_r$ differential decreases the May filtration by $r$, and it changes bidegrees by $(-1,1)$. (In other words, May differentials point one unit left and one unit up.)

(14) To begin, we need that $d_1(h_{02}) = h_{01}h_{12}$. Using this formula, compute that the $E_2$-page is $\mathbb{F}_2[h_{01}, h_{12}, b_{02}]/h_{01}h_{12}$, where $b_{02} = h_{02}^2$.

(15) Use the Massey product shuffle $h_{12}\langle h_{01}, h_{12}, h_{01}\rangle = \langle h_{12}, h_{01}, h_{12}\rangle h_{01}$ to deduce that $h_{12}^3$ must be hit by a differential. Conclude that $d_2(b_{02}) = h_{12}^3$.

(16) Compute that the $E_3$-page is

$$\frac{\mathbb{F}_2[h_{01}, h_{12}, a, b]}{h_{01}h_{12}, h_{12}^3, h_{12}a, a^2 + h_{01}^2 b},$$

where $a = h_{01}b_{02}$ and $b = b_{02}^2$.

(17) Verify that there are no possible higher May differentials.

(18) You have now obtained the $E_2$-page of the Adams spectral sequence for $ko$. Verify that there are no possible Adams differentials, so you have obtained the homotopy of $ko$.

(19) Find a threefold Massey product decomposition for $a$. Find a fourfold Massey product decomposition for $b$.

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