Massey Products

A is a differential graded algebra. (typically not commutative)

Leibniz rule : $d(xy) = dx \cdot y + x \cdot dy$.

Ex: Cobar complex whose homology is $\mathrm{Ext}_A(\mathbb{F}_2, \mathbb{F}_2) =$ Adams E_2-page.

Ex: E_r-page of the May spectral sequence, equipped with the May d_r differential

Ex: E_r-page of the Adams spectral sequence, equipped with the Adams d_r differential.

$H(A)$ inherits a product structure.
well-defined because of the Leibniz rule.

Let a_{01}, a_{12}, a_{23} be cycles in A $\left(d a_{i,i+1} = 0, \ \overline{a_{i,i+1}} \in H(A) \right)$

such that $\overline{a_{01}} \cdot \overline{a_{12}} = 0$ and $\overline{a_{12}} \cdot \overline{a_{23}} = 0$ ($a_{01}a_{12}$ and $a_{12}a_{23}$ are boundaries)

$\overline{a_{01}} \cdot \overline{a_{12}} \cdot \overline{a_{23}} = 0$ for two different reasons.

Choose $a_{02}, a_{13} \in A$ such that $da_{02} = a_{01}a_{12}$, $da_{13} = a_{12}a_{23}$

$\langle \overline{a_{01}}, \overline{a_{12}}, \overline{a_{23}} \rangle = \left\{ \overline{a_{02}a_{23} + a_{01}a_{13}} \right\}$ (all possible choices of a_{02}, a_{13})

$\langle \overline{a_{01}}, \overline{a_{12}}, \overline{a_{23}} \rangle$ is a coset in $H(A)$ because of the choices.

Any two choices differ by a cycle.

The <u>indeterminacy</u> of $\langle \overline{a_{01}}, \overline{a_{12}}, \overline{a_{23}} \rangle$ is $\overline{a_{01}} \cdot H(A) + H(A) \cdot \overline{a_{23}}$.

<u>Ex</u>: $A = \mathbb{F}_2[h_{01}, h_{12}, h_{23}, h_{02}, h_{13}, h_{03}]$

$$d h_{02} = h_{01} h_{12} \qquad h_{01} \quad h_{12} \quad h_{23}$$

$$d h_{13} = h_{12} h_{23} \qquad h_{02} \quad h_{13}$$

$$d h_{03} = h_{01} h_{13} + h_{02} h_{23} \qquad h_{03}$$

$$\langle \overline{h_{01}}, \overline{h_{12}}, \overline{h_{23}} \rangle = \overline{h_{02} h_{23} + h_{01} h_{13}} = \overline{d h_{03}} = \overline{0}.$$

$$h_{02} \quad h_{13}$$

<u>Note</u>: This is the E_1-page of the May spectral sequence that converges to $\mathrm{Ext}_{A(2)}(\mathbb{F}_2, \mathbb{F}_2) = E_2$-page of the Adams spectral sequence for tmf.

<u>Higher Massey products</u>

$$\langle \overline{a_{01}}, \overline{a_{12}}, \overline{a_{23}}, \overline{a_{34}} \rangle$$

$$a_{02} \quad a_{13} \quad a_{24}$$

$$a_{03} \quad a_{14}$$

Need: $\overline{a_{01}}\,\overline{a_{12}} = 0$, $\overline{a_{12}}\,\overline{a_{23}} = 0$, $\overline{a_{23}}\,\overline{a_{34}} = 0$

$0 \in \langle \overline{a_{01}}, \overline{a_{12}}, \overline{a_{23}} \rangle$

$0 \in \langle \overline{a_{12}}, \overline{a_{23}}, \overline{a_{34}} \rangle$

$$\{ \overline{a_{01} a_{14} + a_{02} a_{24} + a_{03} a_{34}} \}$$ (Check that this is a cycle)

<u>Note</u>: Higher brackets are similar. Need all subbrackets to vanish.

$$\langle \overline{a_{01}}, \ldots, \overline{a_{n-1,n}} \rangle = \left\{ \overline{\sum_{0<k<n} a_{0k} a_{kn}} \right\}$$

<u>Defn</u>: The May E_1-page is $\mathbb{F}_2[h_{ij} \mid 0 \le i < j]$ with differential $d h_{ij} = \sum_{i<k<j} h_{ik} h_{kj}$

<u>Notation</u>: Nearly every author has a different notation for these elements. This notation is inspired by May's "Matric Massey products".

$$\begin{array}{ccccc} h_{01} & h_{12} & h_{23} & h_{34} & h_{45} \; \cdots \\ h_{02} & h_{13} & h_{24} & h_{35} \; \cdots & \\ h_{03} & h_{14} & h_{25} \; \cdots & & \\ h_{04} & h_{15} \; \cdots & & & \\ h_{05} \; \cdots & & & & \end{array}$$

<u>Slogan</u>: The May E_1-page is the universal infinite Massey product that contains zero.

Algebra of Massey Products

Massey products satisfy many relations, such as:

$$0 \in \langle a_{01}, a_{12}, a_{23} \rangle \quad \text{if } a_{01} = 0 \text{ or } a_{12} = 0 \text{ or } a_{23} = 0$$

$$\langle a_{01} + a_{01}', a_{12}, a_{23} \rangle \subseteq \langle a_{01}, a_{12}, a_{23} \rangle + \langle a_{01}', a_{12}, a_{23} \rangle$$

$$\langle a_{01}, a_{12}, a_{23} + a_{23}' \rangle \subseteq \langle a_{01}, a_{12}, a_{23} \rangle + \langle a_{01}, a_{12}, a_{23}' \rangle$$

$$\langle a_{01}, a_{12} + a_{12}', a_{23} \rangle = \langle a_{01}, a_{12}, a_{23} \rangle + \langle a_{01}, a_{12}', a_{23} \rangle$$

$$a_{01} \langle a_{12}, a_{23}, a_{34} \rangle \subseteq \langle a_{01} a_{12}, a_{23}, a_{34} \rangle$$

$$\langle a_{01}, a_{12}, a_{23} \rangle a_{34} \subseteq \langle a_{01}, a_{12}, a_{23} a_{34} \rangle$$

$$\langle a_{01} a_{12}, a_{23}, a_{34} \rangle \subseteq \langle a_{01}, a_{12} a_{23}, a_{34} \rangle$$

$$\langle a_{01}, a_{12}, a_{23} a_{34} \rangle \subseteq \langle a_{01}, a_{12} a_{23}, a_{34} \rangle$$

$$a_{01} \langle a_{12}, a_{23}, a_{34} \rangle = \langle a_{01}, a_{12}, a_{23} \rangle a_{34}$$

Assuming that all brackets are defined.

<u>Note</u>: There is no known complete list of such relations for higher brackets.

Massey products in Ext

$\mathrm{Ext}_A(\mathbb{F}_2, \mathbb{F}_2)$ has Massey products because it is the homology of the cobar complex.

Computations in the cobar complex are difficult, even in low dimensions.

<u>Thm</u> [Adams]: $h_{n+1,n+2}\, x \in \langle \overset{(h_n)}{h_{n,n+1}}, x, \overset{(h_n)}{h_{n,n+1}} \rangle$

$\{h_1^2\} = \langle h_0, h_1, h_0 \rangle$

<u>Ex</u>: $h_{12}^2 = \langle h_{01}, h_{12}, h_{01} \rangle$ $\qquad$ $h_{23}^2 = \langle h_{12}, h_{23}, h_{12} \rangle$

$h_{01}h_{23} = \langle h_{12}, h_{01}, h_{12} \rangle$ $\qquad$ $h_{12}h_{34} = \langle h_{23}, h_{12}, h_{23} \rangle$

Use Massey products to deduce May differentials, and use May differentials to deduce Massey products.

<u>May Convergence Theorem</u>: A Massey product $d \in \langle a, b, c \rangle$ in $H(\overset{\text{May}}{E_r})$ implies that there is a corresponding Massey product in Ext. (There are some technical conditions that are not always satisfied.)

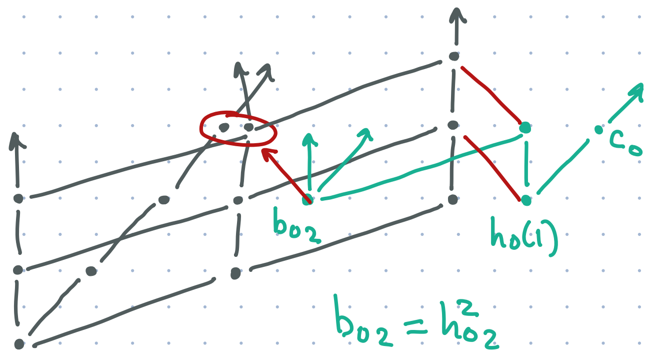
<u>Ex</u>: $\langle h_{12}^2, h_{01}, h_{12}, h_{23} \rangle$

$$\begin{array}{cccc} h_{12}^2 & h_{01} & h_{12} & h_{23} \\ & h_{12}h_{02} & h_{02} & h_{13} \\ & & 0 & h_{03} \end{array}$$

Use the May d_1.

$h_{12}^2 h_{03} + h_{12} h_{02} h_{13} = h_{12}(h_{12} h_{03} + h_{02} h_{13}) = h_{12} h_0(1) = c_0$.

Ex: May E_2-page



① $h_{12}^3 = h_{12} \cdot h_{12}^2 = h_{12}\langle h_{01}, h_{12}, h_{01}\rangle \overset{!}{=} \langle h_{12}, h_{01}, h_{12}\rangle h_{01} = h_{01}h_{23}h_{01} = h_{01}^2 h_{23}$

② Therefore $d_2(b_{02}) = h_{12}^3 + h_{01}^2 h_{23}$

③ Then $d_2(h_{23}b_{02}) = h_{01}^2 h_{23}^2$, so

④ $d_2(h_0(1)) = h_{01}h_{23}^2$

⑤ $c_0 = h_{12}h_0(1) = \langle h_{12}, h_{01}, h_{23}^2\rangle$

Toda brackets in π_*

Toda brackets are another example of rich additional structure.

The name "stable homotopy ring" is an injustice to the depth + intricacy of the structure of π_*.

<u>Moss</u> <u>Convergence</u> <u>Theorem</u>: A Massey product in $\mathrm{Ext}_A(\mathbb{F}_2, \mathbb{F}_2)$, or
Adams
in $H(E_r)$, implies that there is a corresponding Toda bracket in π_*. (There are some technical conditions that are not always satisfied.)

Ex: $c_0 = \langle h_1^2, h_0, h_1, h_2 \rangle$ $\quad\quad$ $\varepsilon = \langle \eta^2, 2, \eta, \nu \rangle$

$c_0 = \langle h_1, h_0, h_2^2 \rangle$ $\quad\quad$ $\varepsilon \in \langle \eta, 2, \nu^2 \rangle = \{\varepsilon, \varepsilon + \eta\sigma\}$

$h_1 h_4 = \langle h_1, h_0, h_3^2 \rangle$ in $E_3 = H(E_2)$ $\quad\quad$ $\eta_4 \in \langle \eta, 2, \sigma^2 \rangle = \{\eta_4, \eta_4 + [Pc_0]\}$