Connective real K-theory ko

$A_* =$ dual Steenrod algebra $= \mathbb{F}_2[\xi_1, \xi_2, \ldots]$

$H_*(H\mathbb{F}_2) = \mathbb{F}_2[\xi_1, \xi_2, \ldots]$

$H\mathbb{Z} \xrightarrow{2} H\mathbb{Z} \longrightarrow H\mathbb{F}_2 \implies H_*(H\mathbb{Z}) = \mathbb{F}_2[\xi_1^2, \xi_2, \ldots]$

ξ_1 is dual to Sq^1

$ku \xrightarrow{v_1} ku \longrightarrow H\mathbb{Z} \implies H_*(ku) = \mathbb{F}_2[\xi_1^2, \xi_2^2, \xi_3, \ldots]$

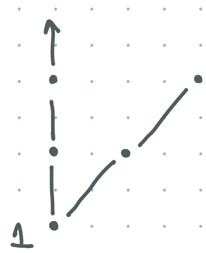
ξ_2 is dual to $Q_1 = Sq^1 Sq^2 + Sq^2 Sq^1$

$ko \xrightarrow{\eta} ko \longrightarrow ku \implies H_*(ko) = \mathbb{F}_2[\xi_1^4, \xi_2^2, \xi_3, \ldots]$

ξ_1^2 is dual to Sq^2

$\pi_* ko \Leftarrow \mathrm{Ext}_{A(1)}(\mathbb{F}_2, \mathbb{F}_2)$, where $A(1)_* = \mathbb{F}_2[\xi_1, \xi_2, \ldots]/\xi_1^4, \xi_2^2, \xi_3, \ldots$

Adams E_∞-page



$\pi_{**}(ko^{\mathbb{C}}) \Leftarrow \mathrm{Ext}_{A(1)^{\mathbb{C}}}(\mathbb{F}_2[\tau], \mathbb{F}_2[\tau])$

Adams filtration

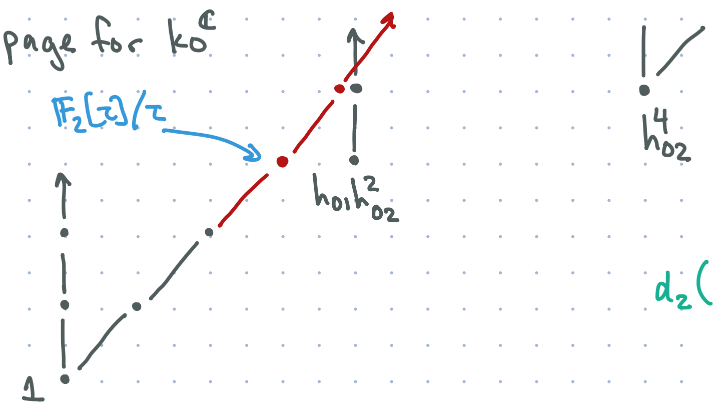
stem

motivic weight

$\mathbb{F}_2[\tau] = \mathbb{C}$-motivic homology of a point, $|\tau| = (0, 0, -1)$

$A(1)^{\mathbb{C}}_* = \mathbb{F}_2[\tau][\tau_0, \tau_1, \xi_1] / \tau_0^2 = \tau\xi_1, \xi_1^2, \tau_1^2$

Adams E_∞-page for $ko^{\mathbb{C}}$



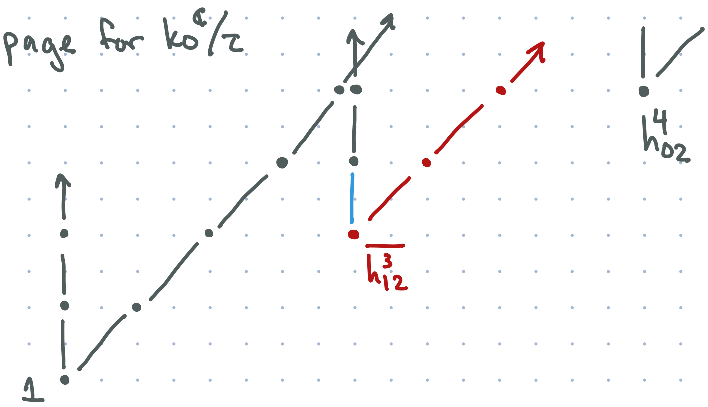
$d_2(b_{02}) = \tau h_{12}^3$

The Adams-Novikov spectral sequence for ko can be reconstructed from the $\mathbb{C}$-motivic Adams spectral sequence for ko.

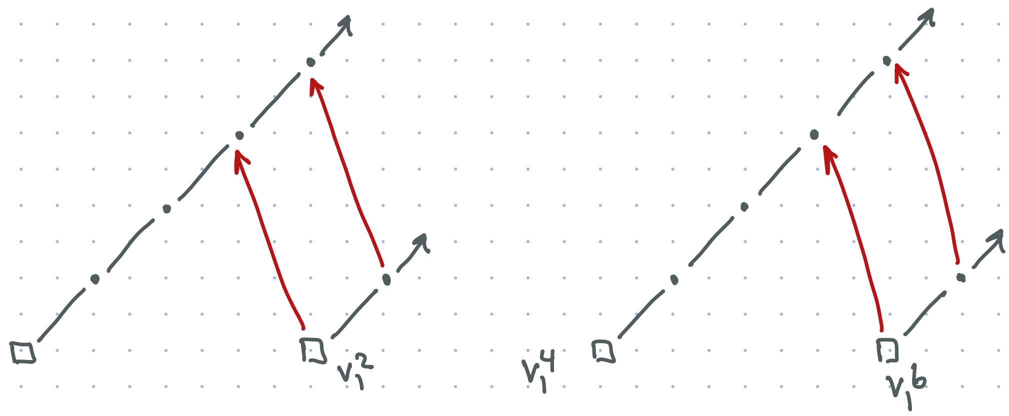
$ko^{\mathbb{C}} \xrightarrow{\tau} ko^{\mathbb{C}} \longrightarrow ko^{\mathbb{C}}/\tau$ yields

$$0 \longrightarrow \operatorname{coker}(\tau) \longrightarrow \operatorname{Ext}_{\mathbb{C}}(ko^{\mathbb{C}}/\tau) \longrightarrow \Sigma \ker(\tau) \longrightarrow 0$$

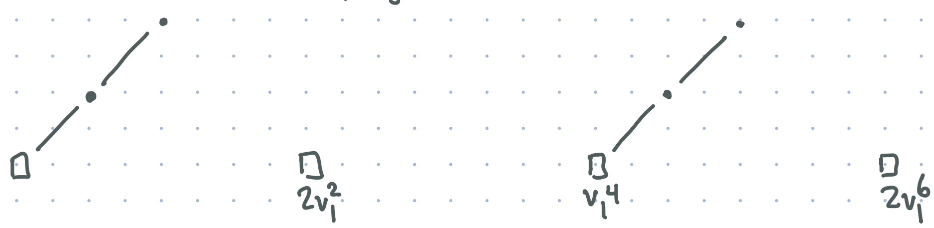
Adams E_∞-page for $ko^{\mathbb{C}}/\tau$



Adams-Novikov E_2-page for ko



Adams-Novikov E_∞-page



Now we need geometric input.

$\psi^3-1 : ko \longrightarrow ko$ induces multiplication by $3^{2k}-1$ on

$$\begin{array}{ccc} \pi_{4k} ko & \longrightarrow & \pi_{4k} ko \\ \mathbb{Z} & \longrightarrow & \mathbb{Z} . \end{array}$$

Define j' to be the fiber of $ko \xrightarrow{\psi^3-1} ko$.

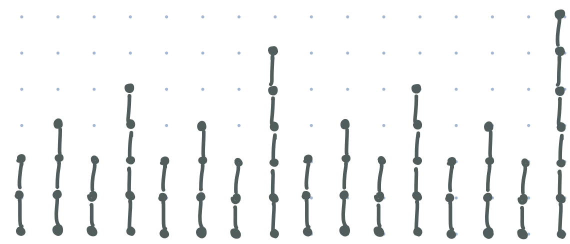
The fiber sequence $j' \to ko \xrightarrow{\psi^3-1} ko$ induces a long exact sequence of Adams-Novikov E_2-pages.

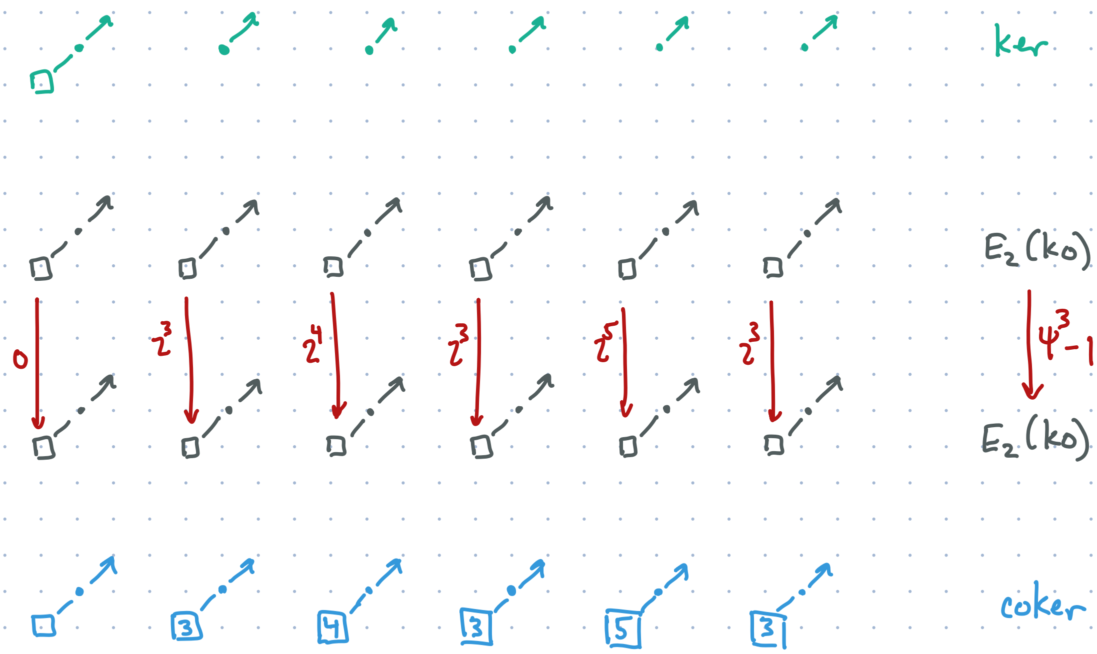
$$0 \longrightarrow \Sigma^{-1} \operatorname{coker}(\psi^3-1) \longrightarrow E_2(j') \longrightarrow \ker(\psi^3-1) \longrightarrow 0$$

$\operatorname{Coker}(\psi^3-1)_{4k} = \mathbb{Z}/2^{v(3^{2k}-1)}$, where $v(3^{2k}-1)$ is 2-adic valuation

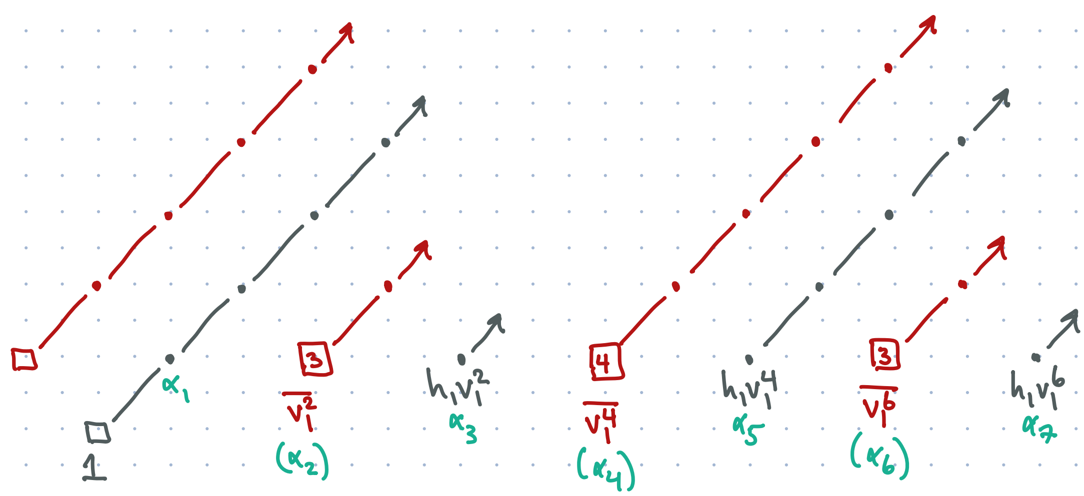
Odd integers are units in our 2-primary situation.

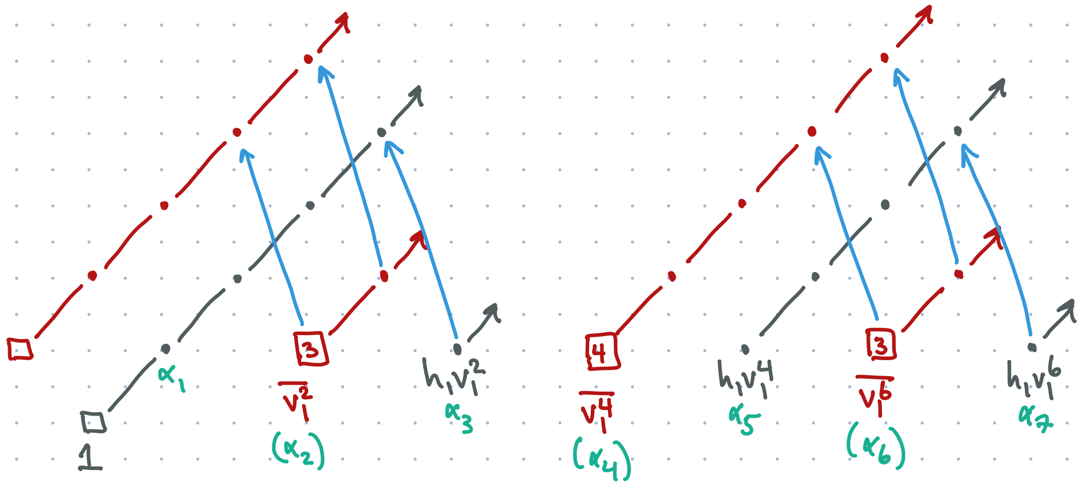
$$v(3^{2k}-1) = \{3, 4, 3, 5, 3, 4, 3, 6, 3, 4, 3, 5, \dots\}_{k=1}^{\infty}$$



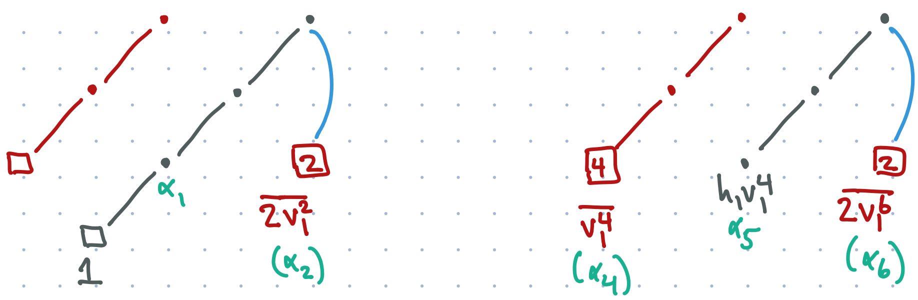


Adams-Novikov E_2-page for j'





Adams–Novikov E_∞-page for j



Thm: Let X be a spectrum, and let $\alpha \in \pi_* X$. Then either

1. $2\alpha \neq 0$, or
2. $\alpha\eta^2$ is divisible by 2.

(or both)

Pf: X is an S-module, so $\pi_* X$ is a "Toda module" over $\pi_* S$.

$\langle \alpha, \beta, \gamma \rangle$ for $\alpha \in \pi_* X$, $\beta, \gamma \in \pi_* S$ such that $\alpha\beta = 0$, $\beta\gamma = 0$.

If $2\alpha = 0$, then

$$\alpha\eta^2 = \alpha\langle 2, \eta, 2\rangle = \langle \alpha, 2, \eta\rangle 2 . \blacksquare$$

Note: One can remove the errors in stems $-1, 0, 1$ by considering

$$j \longrightarrow ko \xrightarrow{\psi^3 - 1} \Sigma^4 ksp .$$