Some chromatic equivariant homotopy theory

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February 24, 2022

First question of the day

Big problem

Compute the G-equivariant stable stems $\pi_\star S_G$ for G a finite group.

We want to take a chromatic approach.

The classical story

Chromatic homotopy theory says S is built from "pieces" $S_{K(n)}$.

1. Nontrivial gluing, coming from the chromatic tower

$$S_{(p)} \simeq \lim_{n\to\infty} L_n S, \qquad L_n S = L_{n-1} S \times_{L_{n-1} S_{K(n)}} S_{K(n)},$$

   but already $S \to S_{K(n)}$ detects a lot.

2. Have a method for computing: $S_{K(n)} = E_n^{\mathrm{h}\mathbb{G}_n}$.
   1. Don't have to go all the way: also have approximations $E_n^{\mathrm{h}K}$.
   2. Have spectral sequences $H_c^*(K; \pi_* E_n) \Rightarrow \pi_* E_n^{\mathrm{h}K}$.

Question

Is there a "G-equivariant $K(n)$-local sphere" giving a similar story?

One possible approach: follow the Segal conjecture

Definition

Write $b_G\colon \mathcal{S}\mathrm{p} \to \mathcal{S}\mathrm{p}_G$ for the Borel / cofree functor.

1. In other notation, $b_G(X) = F(EG_+, i_*X)$.
2. Satisfies $\mathcal{S}\mathrm{p}_G(S^\alpha, b_G(X)) = \mathcal{S}\mathrm{p}(\mathrm{Th}(\alpha), X)$ for $\alpha \in RO(G)$.
3. Here, $\mathrm{Th}(\alpha) = (S^\alpha)_{\mathrm{h}G} =$ Thom spectrum of α over BG.
4. In particular, $\pi_\alpha b_G(X) = X^0\,\mathrm{Th}(\alpha)$.

The Segal conjecture (Carlsson in general; important cases by Lin, Gunawardena, Ravenel, Adams, Miller)

If G is a p-group, then $S_G \simeq b_G(S)$ up to p-completion.

What this tells us

$b_G(S_{K(n)})$ is a fair candidate for a "G-equivariant $K(n)$-local sphere".

Some nice properties of $b_G(S_{K(n)})$

Good approximations to the sphere

The Segal conjecture says $S_G \approx b_G(S)$. As b_G preserves limits, have

$$b_G(S_{(p)}) \simeq \lim_{n\to\infty} b_G(L_n S),$$
$$b_G(L_n S) = b_G(L_{n-1}S) \times_{b_G(L_{n-1}S_{K(n)})} b_G(S_{K(n)}).$$

So you know from the start that $b_G(S_{K(n)})$ see a lot of S_G.

Can compute: coefficients are nonequivariant cohomotopy

1. Equivalences $b_G(S_{K(n)}) = b_G(E_n)^{\mathrm{h}\mathbb{G}_n}$;
2. In general, have $b_G(E_n^{\mathrm{h}K}) = b_G(E_n)^{\mathrm{h}K}$ and

$$H_c^*(K; \pi_\star b_G(E_n)) \Rightarrow \pi_\star b_G(E_n^{\mathrm{h}K}).$$

Acts like a finite spectrum

1. (Hovey, Greenlees, Sadofsky): $\mathcal{S}\mathrm{p}(\mathrm{Th}(\alpha), S_{K(n)}) \simeq L_{K(n)}\,\mathrm{Th}(-\alpha)$.
2. Thus $b_G(S_{K(n)})$ is built from $K(n)$-locally dualizable pieces.

A more conceptual approach

Observation

1. $S_{K(n)}$ pops out of general theory as a Bousfield localization.
2. By contrast, $b_G(S_{K(n)})$ was defined "by hand".

Question

Is there a good G-equivariant $K(n)$, say $K(n)_G$, leading to $L_{K(n)_G} S_G$?

Answer for $n = 1$

Have good G-equivariant K-theories KU_G, so $K(1)_G := KU_G/(p)$.

The G-equivariant $K(n)$-local sphere

Theorem (special case of the Atiyah-Segal completion theorem)

If G is a p-group, then $KU_G/(p) = b_G(KU/(p))$.

That is, $K(1)_G = b_G(K(1))$.

Intuition

If $E_G =$ "KU_G at height n" (e.g. equiv. elliptic cohomology for $n = 2$), then expect $E_G/(p, v_1, \ldots, v_{n-1}) = b_G(K(n))$ for G a p-group.

What this intuition leads us to

$b_G(K(n))$ is a good choice of $K(n)_G$, at least for G a p-group.

Punchline

For any G, have $L_{b_G(K(n))} S_G \simeq b_G(S_{K(n)})$.

So the manual construction and conceptual approach agree.

Towards computations

Things to compute

1. $\pi_\star b_G(E_n) = E_n^0 \,\mathrm{Th}(\star)$, including its Morava stabilizer action;
2. Descent $H^*(K; \pi_\star b_G(E_n)) \Rightarrow \pi_\star b_G(E_n^{\mathrm{h}K})$.

Remark: odd primes

If $|G|$ is odd, then nontrivial irreducible orthogonal G-representations admit complex structures, so $\pi_\star b_G(E_n)$ amounts to $E_n^* BG$.

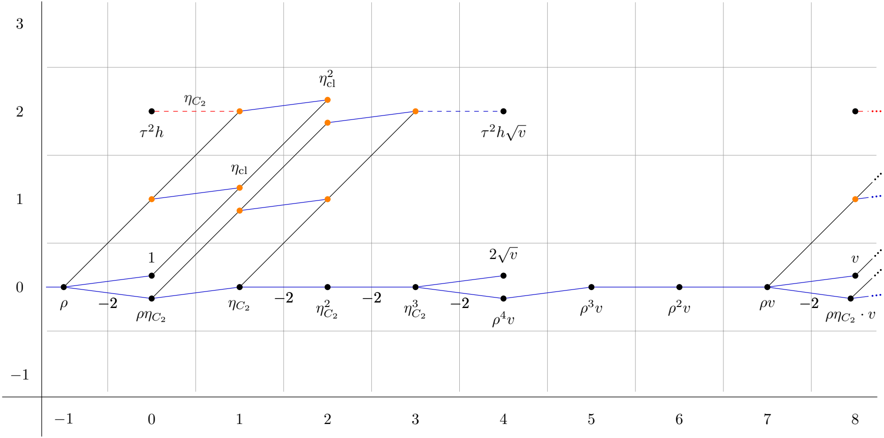
Remark: decompletions

1. Rather than $b_G(E_1)$ and $b_G(E_1^{\mathrm{h}C_2})$, better is KU_G and KO_G.
2. Height 2 analogue: G-equiv. $(TMF$ + adjectives$)$, where defined.

A selection of relevant computations

1. (Karoubi 2002) $\pi_\star KU_G$ as a group for any G.
2. (Hu-Kriz 2006) More on $\pi_\star KU_A$ for $A = C_2^n$.
3. (Guillou-Hill-Isaksen-Ravenel 2019) $\pi_\star (ko_{C_2})_2^\wedge$ via Adams SS.
4. (B. 2021) $\pi_\star b_{C_2}(S_{K(1)})$ as a ring by descent from $\pi_\star b_{C_2}(KU_2^\wedge)$.
5. (B. 2022) $\pi_\star KU_A$ and $\pi_\star KO_A$ with products/transfers/ restrictions/norms/Adams ops for $A \simeq C_2^n$.
6. (Bonventre-Guillou-Stapleton forthcoming) $\pi_0 L_{KU_G} S_G$ for G an odd p-group. Notice: no completion.
7. (B. forthcoming) $\pi_\star b_{C_2}(BPR)$ and $\pi_\star TMF_0(3)_{C_2}$ and friends.

Example: C_2-equivariant real K-theory



1. Coord (s,c) is $\pi_{c+(s-c)\sigma}KO_{C_2}$, black dots are $\mathbb{Z}$, orange are $\mathbb{Z}/(2)$.
2. Blue lines are multiplication by $\rho \in \pi_{-\sigma}S$, the Euler class $S^0 \to S^\sigma$.
3. η_{C_2} is C_2-Hopf map; relation $\rho^2\eta_{C_2} = -2\rho$ encodes $R(2) = \eta$.
4. Dashed lines are hidden in HFPSS $H^*(C_2; \pi_\star KU_{C_2}) \Rightarrow \pi_\star KO_{C_2}$.

Relation to $\mathbb{R}$eal C_2-equivariant homotopy theory

Important note

The "C_2" in KU_{C_2} isn't the C_2 acting on KU.

Theorem (see Guillou-Hill-Isaksen-Ravenel 2019)

Where $\eta_{C_2} \in \pi_\sigma KO_{C_2}$, there is an equivalence $KO_{C_2}/(\eta_{C_2}) \simeq K\mathbb{R}$.

Comments

1. Thus $K\mathbb{R}$ kills all the best stuff in S_{C_2}, like Mahowald invariants.
2. Classically, $KO/(\eta_{\mathrm{cl}}) \simeq KU$, so η_{cl} nilpotent gives $\langle KU \rangle = \langle KO \rangle$. Now: Bousfield classes $\langle KU_{C_2} \rangle = \langle KO_{C_2} \rangle \subsetneq \langle K\mathbb{R} \rangle \subsetneq \langle b_{C_2}(K(1)) \rangle$.

This theorem is fairly generic. For example:

Theorem (B.)

1. There is $\xi \in \pi_\sigma b_{C_2}(MUP^{\mathrm{h}C_2})$ with $b_{C_2}(MUP^{\mathrm{h}C_2})/(\xi) \simeq MUP\mathbb{R}$.
2. $\zeta \in \pi_{17\sigma} TMF_0(3)_{C_2}$ with $TMF_0(3)_{C_2}/(\zeta) \simeq (TMF_1(3) \circlearrowleft C_2)$.