eCHT Minicourse
The Slice Spectral Sequence
Problem Set 1
Spring 2022

1. Let $\overline{\rho} = \rho - 1$ denote the reduced regular representation of G, and let $a_{\overline{\rho}}\colon S^0 \hookrightarrow S^{\overline{\rho}}$ be the inclusion of the fixed points. Show that the colimit

$$S^{\infty\overline{\rho}} = \mathrm{colim}(S^0 \xrightarrow{a_{\overline{\rho}}} S^{\overline{\rho}} \xrightarrow{a_{\overline{\rho}}} S^{2\overline{\rho}} \xrightarrow{a_{\overline{\rho}}} \dots)$$

is a model for the G-space $\widetilde{E\mathcal{P}}$. (Hint: What is the restriction $\downarrow_H^G \rho$ for H a proper subgroup?)

As a consequence, the $RO(G)$-graded homotopy groups of a geometric fixed point spectrum $\Phi^G X$ can be obtained from those of X by inverting the element $a_{\overline{\rho}}$.

2. Display the lattice of subgroups (and corresponding Weyl groups) for the following groups:

   (a) $G = C_p$,

   (b) $G = C_{p^2}$,

   (c) $G = C_2 \times C_2$,

   (d) $G = C_3 \times C_3$,

   (e) $G = C_6$, and

   (f) $G = D_3$, the dihedral group of order 6.

3. Given a subgroup $H \leq G$ and an H-Mackey functor $\underline{M}$, there is a G-Mackey functor $\uparrow_H^G \underline{M}$, known as the induced Mackey functor. One way to describe this is by using the alternate characterization of G-Mackey functors as indexed over finite G-sets, rather than just the G-orbits. Then, for a finite G-set X, the value of $\uparrow_H^G \underline{M}$ at X is the value of $\underline{M}$ at the H-set $\downarrow_H^G X$.

   Determine the following induced Mackey functors, including the actions of the Weyl groups.

   (a) $\uparrow_e^{C_p} \mathbb{Z}$.

   (b) $\uparrow_e^{C_{p^2}} \mathbb{Z}$.

   (c) $\uparrow_{C_2}^{C_4} \underline{M}$, for $\underline{M} \in \mathrm{Mack}(C_2)$.

   (d) $\uparrow_{C_2}^{C_2 \times C_2} \underline{M}$, for $\underline{M} \in \mathrm{Mack}(C_2)$.

4. In the slice spectral sequence for $k\mathbb{R}$, there was an extension problem left to solve. Namely, from the slice spectral sequence, we get an extension of Mackey functors

$$\underline{g} \hookrightarrow \underline{\pi}_2(k\mathbb{R}) \twoheadrightarrow \underline{\mathbb{Z}}^{\sigma}.$$

For any C_2-spectrum X, the transfer map for the Mackey functor $\underline{\pi}_n(X)$ fits into an exact sequence

$$\pi_n^e(X) \to \pi_n^{C_2}(X) \to \pi_n^{C_2}(\Sigma^{\sigma} X).$$

Use this to determine the Mackey functor $\underline{\pi}_2(k\mathbb{R})$.

5. We computed that the nontrivial homotopy Mackey functors of $\Sigma^{\rho} H_{C_2}\underline{\mathbb{Z}}$ are

$$\underline{\pi}_n(\Sigma^{\rho} H_{C_2}\underline{\mathbb{Z}}) \cong \begin{cases} \underline{\mathbb{Z}}^{\sigma} & n = 2 \\ \underline{g} & n = 1. \end{cases}$$

This corresponds to the existence of a fiber sequence

$$\Sigma^2 H_{C_2}\underline{\mathbb{Z}}^{\sigma} \longrightarrow \Sigma^{\rho} H_{C_2}\underline{\mathbb{Z}} \longrightarrow \Sigma^1 H_{C_2}\underline{g}.$$

(a) Compute the homotopy Mackey functors of $\Sigma^{2\rho} H_{C_2}\underline{\mathbb{Z}}$ by showing that $\Sigma^{\rho} H_{C_2}\underline{g} \simeq \Sigma^1 H_{C_2}\underline{g}$ and that $\Sigma^{\rho} H_{C_2}\underline{\mathbb{Z}}^{\sigma} \simeq \Sigma^2 H_{C_2}\underline{\mathbb{Z}}$.

(b) Use induction to compute the homotopy Mackey functors of $\Sigma^{n\rho} H_{C_2}\underline{\mathbb{Z}}$, for $n \geq 0$.

(c) We also saw that $\Sigma^{-\rho} H_{C_2}\underline{\mathbb{Z}} \simeq \Sigma^{-2} H_{C_2}\underline{\mathbb{Z}}^{\sigma}$. Use the fiber sequence

$$\Sigma^{-\sigma} X \longrightarrow X \longrightarrow C_{2+} \wedge X$$

to inductively compute the homotopy Mackey functors of $\Sigma^{-n\rho} H_{C_2}\underline{\mathbb{Z}}$.

6. Show that $\Sigma^n H_{C_2}\underline{\mathbb{Z}}$ is an n-slice for $n = 0, \ldots, 6$.

7. For $G = C_2$, find the 1-slice $P_1^1(S^1)$.