Ravenel-Wilson Hopf Ring Methods in C_2-equivariant Homotopy Theory and the $H\underline{\mathbb{F}}_2$-homology of C_2-equivariant Eilenberg-MacLane spaces

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Hopf rings

Definition

A Hopf ring is an object in the category of graded counital coassociative cocomutative coalgebras.

As such, a Hopf ring consists of

- A sequence of abelian groups $\{M_i\}$
- with a coproduct $\phi(m) = \sum m' \otimes m''$
- and a multiplication $\circ : M_k \otimes M_n \to M_{k+n}$.

Hopf rings & Ω-spectra

Consider an Ω-spectrum

$$G = \{G_k\}.$$

This represents a generalized cohomology theory with

$$G^* X \simeq [X, G_*].$$

Since $G^k X$ is an abelian group, G_k is a H-space.
This H-space structure,

$$* : G_k \times G_k \to G_k,$$

gives rise to a product in homology

$$* : E_* G_k \otimes E_* G_k \to E_*(G_k \times G_k) \to E_* G_k.$$

If $E_*(-)$ has a Künneth isomorphism, $E_*(G_k)$ is a coalgebra and $E_* G_*$ is a Hopf algebra

Hopf rings & Ω-spectra

Suppose G is a ring spectrum.

Then $G^*X \simeq [X, G_*]$ is a graded ring and the multiplication

$$G^k X \times G^n X \to G^{k+n} X$$

has a corresponding multiplication in G_* :

$$\circ : G_k \times G_n \to G_{k+n}$$

and applying $E_*(-)$ we have

$$\circ : E_* G_k \otimes_{E_*} E_* G_n \to E_* G_{k+n},$$

making $E_* G_*$ a Hopf ring.

Ravenel-Wilson Hopf ring method

Idea (Ravenel-Wilson)

The two products $*$ and $\circ$ allow for the construction of many elements from just a few.

Applications (Ravenel-Wilson):

- The mod p homology of classical Eilenberg-MacLane spaces
- The Hopf ring for complex cobordism
- The Morava K-theory of Eilenberg-MacLane spaces

Eilenberg-MacLane Spaces

Consider the Eilenberg-MacLane spectrum for $\mathbb{F}_p$,

$$H\mathbb{F}_p = \{K(\mathbb{F}_p, n)\} = \{K_n\}.$$

Up to homotopy, $H\mathbb{F}_p$ is characterized by

$$H^n(X; \mathbb{F}_p) = [X, K_n].$$

$H_*(-; \mathbb{F}_p)$ has a Künneth isomorphism so $H_* K_*$ is a Hopf ring.

Ravenel-Wilson computational tools

Let Δ^n denote the topological simplex

$$\Delta^n = \{(t_1, t_2, \cdots, t_n) \in \mathbb{R}^n | -1 \leq t_1 \leq \cdots \leq t_n \leq 1\}.$$

The Classical Bar Construction. For a topological monoid A, the pointed space BA is defined as a quotient

$$BA = \coprod_n \Delta^n \times A^{\times n} / \sim$$

Bar Spectral Sequence

The bar construction BA is filtered by

$$B^{[t]}A \simeq \coprod_{t \geq n \geq 0} \Delta^n \times A^n / \sim \qquad \subset BA$$

with associated graded pieces

$$B^{[t]}A / B^{[t-1]}A \simeq S^t \wedge A^{\wedge t}.$$

Applying $H_*(-)$ to these filtered spaces gives the bar spectral sequence with E^1-page

$$E^1_{t,*} = H(S^t) \otimes H_*(A)^{\otimes t}$$

computing $H_*(BA)$.

Ravenel-Wilson computational tools

The bar spectral sequence has

$$E^2_{*,*} \simeq \mathrm{Tor}^{H_*K_m}_{*,*}(\mathbb{F}_p, \mathbb{F}_p) \Longrightarrow H_*BK_m \cong H_*K_{m+1}.$$

Useful homological algebra:

$$\mathrm{Tor}^{E[x]}(\mathbb{F}_p, \mathbb{F}_p) \simeq \Gamma[e_1 x]$$

$$\mathrm{Tor}^{T[x]}(\mathbb{F}_p, \mathbb{F}_p) \simeq E[e_1 x] \otimes \Gamma[\phi x]$$

Circle product structure

The cup product is induced by a map

$$\circ = \circ_{m,\ell} : K_m \wedge K_\ell \to K_{m+\ell}. \tag{1}$$

The map (1) can be constructed inductively on m.

Assume $\circ_{m,\ell}$ has been defined,

Replace K_{m+1} and $K_{m+\ell+1}$ with bar constructions

$$\{\coprod_n \Delta^n \times K_m^n / \sim\} \wedge K_\ell \to \{\coprod_n \Delta^n \times K_{m+\ell} / \sim\}. \tag{2}$$

Circle product structure

Define

$$\{\coprod_n \Delta^n \times K_m^n / \sim\} \wedge K_\ell \to \{\coprod_n \Delta^n \times K_{m+\ell} / \sim\}. \tag{2}$$

by

$$\{(t, x)\} \circ y = \{(t, x \circ y)\} \tag{3}$$

where $t \in \Delta^n$, $x = (x_1, \cdots, x_n) \in K_m$, and $y \in K_\ell$.

Theorem (Ravenel-Wilson)

The above construction is well defined and gives the cup product pairing

$$\circ : BK_m \wedge K_\ell \to BK_{m+\ell}.$$

Theorem (Thomason-Wilson)

The $\circ$ product factors as

$$\begin{array}{ccc} B^{[t]}K_m \times K_\ell & \to & B^{[t]}K_{m+\ell} \\ \bigcap & & \bigcap \\ \circ : BK_m \times K_\ell & \to & BK_{m+\ell} \end{array}$$

and the map

$$\begin{array}{ccc} B^{[t]}K_m / B^{[t-1]}K_m \times K_\ell & \to & B^{[t]}K_{m+\ell} / B^{[t-1]}K_{m+\ell} \\ \wr| & & \wr| \\ S^t \wedge K_m^{\wedge t} \times K_\ell & \to & S^t \wedge K_{m+\ell}^{\wedge t} \end{array}$$

is described inductively as $(k_1, \cdots, k_t) \circ k = (k_1 \circ k, \cdots, k_t \circ k)$.

Theorem (Thomason-Wilson)

Let $E^r_{*,*}(E_*K_m) \implies E_*K_{m+1}$ be the bar spectral sequence. Compatible with

$$\circ : E_*K_{m+1} \otimes_{H_*} E_*K_\ell \to E_*K_{m+\ell+1}$$

is a pairing

$$E^r_{t,*}(E_*K_m) \otimes_{H_*} E_\star K_\ell \to E^r_{t,*}(E_*K_{m+\ell})$$

with $d^r(x) \circ y = d^r(x \circ y)$. For $r = 1$ this pairing is given by

$$(k_1|\cdots|k_t) \circ k = \sum \pm(k_1 \circ k'|k_2 \circ k''|\cdots|k_s \circ k^{(t)})$$

where $k \to \sum k' \otimes k'' \otimes \cdots \otimes k^{(t)}$ is the iterated reduced coproduct.

Ravenel-Wilson computational tools

Example

$$H_*K_1 \cong E[e_1] \otimes T[\alpha_{(i)}]$$

$$Tor^{H_*K_1}(\mathbb{F}_p, \mathbb{F}_p) \cong \Gamma[e_2] \otimes E[e_1\alpha_{(i)}] \otimes \Gamma[\phi\alpha_{(i)}]$$

$$\Rightarrow H_*K_2 \cong T[\gamma_{p^i}(e_2)] \otimes E[e_1 \circ \alpha_{(i)}] \otimes T[\alpha_{(i_1)} \circ \alpha_{(i_2)}]$$

Upshot

We can inductively deduce the homology of Eilenberg-MacLane spaces using standard homological algebra!

Mod p homology of Eilenberg-MacLane Spaces

Let $\quad e_1 \in H_1 K_1, \quad \alpha_i \in H_{2i} K_1, \quad \beta_i \in H_{2i} \mathbb{C}P^\infty \quad i \geq 0.$

The generators are $\quad e_1, \quad \alpha_{(i)} = \alpha_{p^i} \quad \beta_{(i)} = \beta_{p^i}.$

For finite sequences

$$I = (i_1, i_2, \cdots), \quad 0 \leq i_1 < i_2 < \cdots,$$
$$J = (j_0, j_1, \cdots), \quad j_k \geq 0,$$

define

$$\alpha_I = \alpha_{(i_1)} \circ \alpha_{(i_2)} \circ \cdots,$$
$$\beta^J = \beta_{(0)}^{\circ j_0} \circ \beta_{(1)}^{\circ j_1} \circ \cdots.$$

Theorem (Ravenel-Wilson)

$$H_* K_* \simeq \otimes_{I,J} E(e_1 \circ \alpha_I \circ \beta^J) \otimes_{I,J} T(\alpha_I \circ \beta^J)$$

Mod 2 homology of Eilenberg-Maclane spaces

For finite sequences

$$I = (i_{(-1)}, i_0, i_1, i_2, \cdots), \qquad\qquad i_k \geq 0$$

define

$$(e_1\alpha)^I = e_1^{\circ i_{(-1)}} \circ \alpha_{(0)}^{\circ i_0} \circ \alpha_{(1)}^{\circ i_1} \circ \cdots .$$

Theorem (Ravenel-Wilson)

Then

$$H_*K_* \simeq \otimes_I E[(e_1\alpha)^I]$$

as an algebra where the tensor product is over all I and the coproduct follows by Hopf ring properties form the α's.

Stabilizing, the Steenrod algebra

Homology suspend $\beta_{(i)}$ to define

$$\xi_i \in H_{2(p^i-1)}H,$$

Homology suspend $\alpha_{(i)}$ to define

$$\tau_i \in H_{2p^i-1}H.$$

Then

$$H_*H \simeq E[\tau_0, \tau_1, \cdots] \otimes P[\xi_1, \xi_2, \cdots].$$

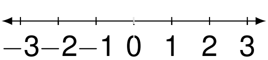
$RO(G)$-graded homology

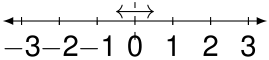
G a compact Lie group

- V a real representation of G, S^V a representation sphere
- (Co)homology graded on the real representation ring $RO(G)$

$G = C_2$

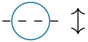
- Two irreducible representations $\mathbb{R}_{triv}$ and $\mathbb{R}_{sign}$ (also denoted σ)





- Representation spheres S^1 and S^σ

S^1 

S^σ 

- $H_\star(X) = H_\star^{C_2}(X; \mathbb{F}_2)$

$RO(C_2)$-graded homology of a point

$$H_\star(pt, \underline{\mathbb{F}_2}) = \mathbb{F}_2[a,u] \oplus \frac{\mathbb{F}_2[a,u]}{(a^\infty, u^\infty)}\{\theta\}$$

where $|a| = -\sigma$, $|u| = 1 - \sigma$, and $|\theta| = 2\sigma - 2$.

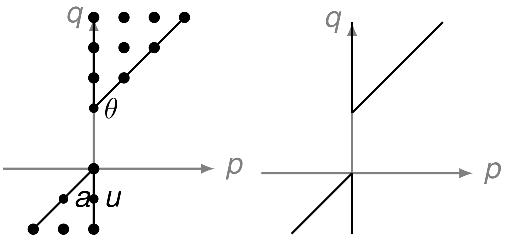


Figure: $H_\star(pt, \underline{\mathbb{F}_2})$ with axis gradings determined by $V \simeq \mathbb{R}^{p-q} \oplus \mathbb{R}^{q\sigma}$.

C_2-Equivariant Eilenberg-MacLane Spaces

The Eilenberg-MacLane spectrum for the C_2-constant Mackey functor $\underline{\mathbb{F}}_2$,

$$H\underline{\mathbb{F}}_2 = \{K(\underline{\mathbb{F}}_2, V)\}_{V \cong k\sigma + l} = \{K_V\}_{V \cong k\sigma + l}.$$

Up to C_2-equivariant homotopy, $H\underline{\mathbb{F}}_2$ is characterized by

$$H^V(X; \underline{\mathbb{F}}_2) = [X, K_V]$$

naturally for all X.

Strategy & equivariant computational tools

- Give explicit models for

  $$* : K_V \times K_V \to K_V$$

  and

  $$\circ : K_V \times K_W \to K_{V+W}$$

  using bar and twisted bar constructions

- Investigate $RO(C_2)$-graded (twisted) bar spectral sequences
  - Recover examples: $H_\star K_\sigma$, $H_\star K(\underline{\mathbb{Z}}, \rho)$
  - Compute: $H_\star K(\underline{\mathbb{Z}}, 2\sigma)$

Strategy & equivariant computational tools

Lemma (Behrens-Wilson)

Suppose $X \in Sp^{C_2}$ and $\{b_i\}$ is a set of elements of $H_\star(X)$ such that

(1) $\{\Phi^e(b_i)\}$ *is a basis of* $H_*(X^e)$ *and*

(2) $\{\Phi^{C_2}(b_i)\}$ *is a basis of* $H_*(X^{\Phi C_2})$,

then $H_\star(X)$ is free over $H_\star$ and $\{b_i\}$ is a basis.

Computation strategy:

- Use $\circ$-products to produce elements in $H_\star K_V$
- Then use explicit models for

$$* : K_V \times K_V \to K_V \qquad \text{and} \qquad \circ : K_V \times K_W \to K_{V+W}$$

to analyze underlying and fixed point maps

Twisted monoids

Definition (Liu)

A C_2-space A is a twisted monoid if it is a topological monoid in the non-equivariant sense with the product satisfying $\gamma(xy) = \gamma(y)\gamma(x)$ where $C_2 \simeq < \gamma >$.

Twisted bar construction

$$B^\sigma A = \amalg_n \Delta^n \times A^n / \sim$$

Twisted Bar Spectral Sequence

The twisted bar construction $B^\sigma A$ is filtered by

$$B^\sigma_t A \simeq \coprod_{t \geq n \geq 0} \Delta^n \times A^n / \sim \qquad \subset B^\sigma A$$

with associated graded pieces

$$B^\sigma_t A / B^\sigma_{t-1} A \simeq S^{\lceil \frac{t}{2} \rceil \sigma + \lfloor \frac{t}{2} \rfloor} \wedge A^{\wedge t}$$

where the C_2-action on A^t is given by

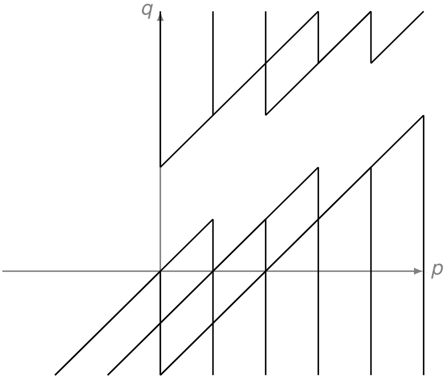
$$\gamma(a_1 \wedge \cdots \wedge a_n) = (\gamma a_n \wedge \cdots \wedge \gamma a_1).$$

Applying $H_\star(-)$ to these filtered spaces gives the twisted bar spectral sequence computing $H_\star(B^\sigma A)$.

Example

$H_\star B^\sigma K_0 \cong H_\star K_\sigma \cong H_\star \mathbb{R}P^\infty_{tw}$ is a free $H_\star$-module with a single generator in each degree $\lceil \frac{n}{2} \rceil \sigma + \lfloor \frac{n}{2} \rfloor$.

Proof: $E^1_{t,\star} \cong H_\star(S^{\lceil \frac{t}{2} \rceil \sigma + \lfloor \frac{t}{2} \rfloor}) \wedge H_\star(\mathbb{F}_2)^{\wedge t}$.



$V \simeq \mathbb{R}^{p-q} \oplus \mathbb{R}^{q\sigma}$

- Filtration degree corresponds to topological degree

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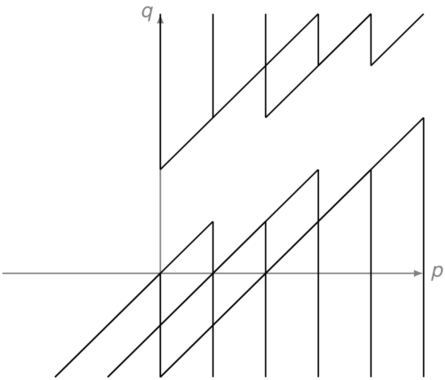

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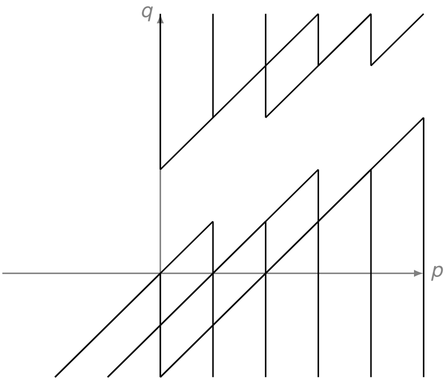

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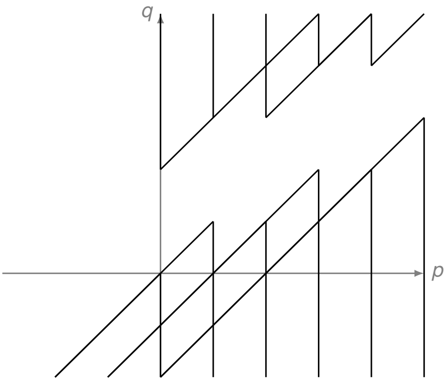


$V \simeq \mathbb{R}^{p-q} \oplus \mathbb{R}^{q\sigma}$

- Filtration degree corresponds to topological degree
- d^r shifts topological degree down by one
- There are no nonzero d^r, $r > 1$
- There are no nonzero d^1, otherwise we would be killing a known generator of $H_*(\mathbb{R}P^\infty)$

The homology of K_σ

$H_\star K_\sigma$ is an exterior algebra on generators

$$e_\sigma, \qquad \bar{\alpha}_{(i)} = \bar{\alpha}_{2^i} \qquad (i \geq 0),$$

where

$$e_\sigma \in H_\sigma K_\sigma, \qquad \bar{\alpha}_i \in H_{\rho i} K_\sigma, \qquad (i \geq 0),$$

and has coproduct

$$\psi(e_\sigma) = 1 \otimes e_\sigma + e_\sigma \otimes 1 + a(e_\sigma \otimes e_\sigma)$$
$$\psi(\bar{\alpha}_n) = \sum_{i=0}^{n} \bar{\alpha}_{n-i} \otimes \bar{\alpha}_i + \sum_{i=0}^{n-1} u(e_\sigma \bar{\alpha}_{n-1-i} \otimes e_\sigma \bar{\alpha}_i)$$

with $a \in H\underline{\mathbb{F}}_{2\{-\sigma\}}$ and $u \in H\underline{\mathbb{F}}_{2\{1-\sigma\}}$.

Collapsing twisted bar spectral sequences

Examples

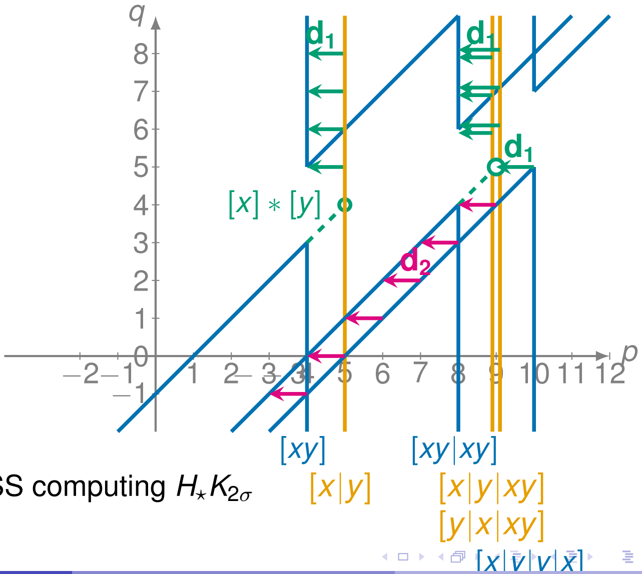
$$H_\star \mathbb{R}P^\infty_{tw} = E[e_\sigma, \bar{\alpha}_{(0)}, \bar{\alpha}_{(1)}, \cdots] = E[e_\sigma] \otimes \Gamma[\bar{\alpha}_{(0)}],\ |e_\sigma| = \sigma,\ |\bar{\alpha}_{(i)}| = 2^i \rho,$$
$$H_\star \mathbb{C}P^\infty_{tw} = E[\bar{\beta}_{(0)}, \bar{\beta}_{(1)}, \cdots] = \Gamma[e_\rho] \text{ where } |\bar{\beta}_{(i)}| = 2^i \rho.$$

Theorem

We have

$$H_\star K(\underline{\mathbb{Z}}, 2\sigma) = E[e_{2\sigma}] \otimes \Gamma[\bar{x}_{(0)}] \text{ where } |e_{2\sigma}| = 2\sigma,\ |\bar{x}_{(0)}| = 2\rho.$$

Differentials in the twisted bar spectral sequence



Twisted bar SS computing $H_\star K_{2\sigma}$

Fixed points of equivariant Eilenberg-MacLane spaces

Proposition (Caruso)

$$(K_{m\sigma+n})^{C_2} \simeq K_n \times \cdots \times K_{n+m}$$

Example

$$(K_\sigma)^{C_2} \simeq K_0 \times K_1$$

Fixed points of equivariant Eilenberg-MacLane spaces

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$$(K_{m\sigma+n})^{C_2} \simeq K_n \times \cdots \times K_{n+m}$$

Example

$$(K_\sigma)^{C_2} \simeq K_0 \times K_1$$

Example (Maps to underlying and fixed point homology)

$H_\star K_\sigma \cong E[e_\sigma, \bar{\alpha}_{(i)}]$

The homology of $K_{*\sigma}$

For finite sequences

$$J = (j_\sigma, j_0, j_1, \cdots) \qquad j_k \geq 0,$$

define

$$(e_\sigma \bar{\alpha})^J = e_\sigma^{\circ j_\sigma} \circ \bar{\alpha}_{(0)}^{\circ j_0} \circ \bar{\alpha}_{(1)}^{\circ j_1} \cdots$$

Theorem

Then

$$H_\star K_{*\sigma} \cong \otimes_J E[(e_\sigma \bar{\alpha})^J]$$

As an algebra where the tensor product is over all J and the coproduct follows by properties of the $\circ$-product from the $\bar{\alpha}$'s.

Theorem (Equivariant Thomason-Wilson)

The $\circ$ product factors as

$$\begin{array}{ccc} B_tK_V \times K_W & \to & B_tK_{V+W} \\ \bigcap & & \bigcap \\ \circ : BK_V \times K_W & \to & BK_{V+W} \end{array}$$

and the map

$$\begin{array}{ccc} {}^{B_tK_V}\!/_{B_{t-1}K_V} \times K_W & \to & {}^{B_tK_{V+W}}\!/_{B_{t-1}K_{V+W}} \\ \wr| & & \wr| \\ S^t \wedge K_V^{\wedge t} \times K_W & \to & S^t \wedge K_{V+W}^{\wedge t} \end{array}$$

is described inductively as $(k_1, \cdots, k_t) \circ k = (k_1 \circ k, \cdots, k_t \circ k)$.

Theorem (Equivariant Thomason-Wilson)

Let $E^r_{*,\star}(E_\star K_V) \implies E_\star K_{V+\sigma}$ be the bar spectral sequence and suppose E^r is $E_\star-$flat for $i \leq r$. Compatible with

$$\circ : E_\star K_{V+1} \otimes_{H_\star} E_\star K_W \to E_\star K_{V+W+1}$$

is a pairing

$$E^r_{t,\star}(E_\star K_V) \otimes_{H_\star} E_\star K_W \to E^r_{t,\star}(E_\star K_{V+W})$$

with $d^r(x) \circ y = d^r(x \circ y)$. For $r = 1$ this pairing is given by

$$(k_1|\cdots|k_t) \circ k = \sum \pm(k_1 \circ k'|k_2 \circ k''|\cdots|k_s \circ k^{(t)})$$

where $k \to \sum k' \otimes k'' \otimes \cdots \otimes k^{(t)}$ is the iterated reduced coproduct.

The homology of K_ρ

Let

$$\bar{\beta}_i \in H_{\rho i} K(\underline{\mathbb{Z}}, \rho), \qquad (i \geq 0).$$

This gives additional generators,

$$\bar{\beta}_{(i)} = \bar{\beta}_{2^i} \quad (i \geq 0),$$

of $H_\star K_\rho$ with coproduct

$$\psi(\bar{\beta}_n) = \sum_{i=0}^{n} \bar{\beta}_{n-i} \otimes \bar{\beta}_i.$$

Theorem

We have

$$H_\star K_\rho \cong E[e_1 \circ \bar{\alpha}_{(i)}, \alpha_{(i_1)} \circ \bar{\alpha}_{(i_2)}, \bar{\beta}_{(i)}]$$

where $i_1 < i_2$ and the coproduct follows by properties of the $\circ$-product from the $\bar{\alpha}_{(i)}$'s and $\bar{\beta}_{(i)}$'s.

Notation for the homology of $K_{\sigma+*}$

Then for finite sequences

$$I = (i_1, i_2, \cdots, i_k), \qquad 0 \le i_1 < i_2 < \cdots,$$

$$W = (w_1, w_2, \cdots, w_q), \qquad 0 \le w_1 < w_2 < \cdots,$$

$$J = (j_{-1}, j_0, j_1, \cdots, j_\ell), \qquad \text{where } j_{-1} \in \{0, 1\} \text{ and all other } j_n \ge 0,$$

and

$$Y = (y_{-1}, y_0, y_1, \cdots, y_r), \qquad \text{where } y_{-1} \in \{0, 1\} \text{ and all other } y_n \ge 0,$$

define

$$(e_1\alpha\beta)^{I,J} = e_1^{\circ j_{-1}} \circ \alpha_{(i_1)} \circ \alpha_{(i_2)} \circ \cdots \circ \alpha_{(i_k)} \circ \beta_{(0)}^{\circ j_0} \circ \beta_{(1)}^{\circ j_1} \circ \cdots \circ \beta_{(\ell)}^{\circ j_\ell},$$

$$(e_1\alpha\beta)^{W,Y} = e_1^{\circ y_{-1}} \circ \alpha_{(w_1)} \circ \alpha_{(w_2)} \circ \cdots \circ \alpha_{(w_q)} \circ \beta_{(0)}^{\circ y_0} \circ \beta_{(1)}^{\circ y_1} \circ \cdots \circ \beta_{(r)}^{\circ j_r},$$

$$|I| = k, \qquad |W| = q \qquad ||J|| = \Sigma j_n, \qquad \text{and } ||Y|| = \Sigma y_n.$$

The homology of $K_{\sigma+*}$

Theorem

Then, we have

$$H_\star K_{\sigma+i} \cong E[(e_1\alpha\beta)^{I,J} \circ \bar{\alpha}_{(m)}, (e_1\alpha\beta)^{W,Y} \circ \bar{\beta}_{(t)}]$$

where $m > i_k$ *and* $m \geq \ell$, $t > w_q$ *and* $t \geq y_r$, $|I| + 2||J|| = i$ *and* $|W| + 2||Y|| = i - 1$, *and the coproduct follows by Hopf ring properties from the* $\alpha_{(i)}$*'s,* $\beta_{(i)}$*'s,* $\bar{\alpha}_{(i)}$*'s and* $\bar{\beta}_{(i)}$*'s.*

The homology of $K_{2\sigma+1}$

Example

$$K_{2\sigma+1} \qquad K^{e}_{2\sigma+1} \simeq K_3 \qquad K^{C_2}_{2\sigma+1} \simeq K_1 \times K_2 \times K_3$$

$H_\star K_{2\sigma+1}$ is exterior on generators

$$e_1 \circ e_\sigma \circ e_\sigma, \qquad \bar{\beta}_{(j_1)} \circ \bar{\alpha}_{(j_2)}, \qquad e_1 \circ \bar{\alpha}_{(j_1)} \circ \bar{\alpha}_{(j_2)},$$

$$\phi^{(k)}(e_\sigma \circ e_\sigma), \qquad \alpha_{(i_1)} \circ \bar{\alpha}_{(i_2)} \circ \bar{\alpha}_{(j_3)}.$$

The homology of $K_{2\sigma+i,\, i\geq 2}$

$K_{2\sigma+i}$ $\qquad$ $K^e_{2\sigma+i} \simeq K_{2+i}$ $\qquad$ $K^{C_2}_{2\sigma+i} \simeq K_i \times K_{i+1} \times K_{i+2}$

Theorem

The homology $H_\star K_{2\sigma+i}$ where $i \geq 2$ is exterior on generators

$$H_*K_{i-2} \circ \bar\beta_{(j_1)} \circ \bar\beta_{(j_1)}, \qquad H_*K_{i-1} \circ \bar\beta_{(j_1)} \circ \bar\alpha_{(j_2)}, \quad H_*K_i \circ e_1 \circ \bar\alpha_{(j_1)} \circ \bar\alpha_{(j_2)},$$

$$H_*K_{i-1} \circ \phi^{(k)}(e_\sigma \circ e_\sigma),$$

The homology of K_V where $2\sigma + 1 \subset V$

Theorem

The $RO(C_2)$-graded homology of K_V, $2\sigma + 1 \subset V$, is exterior on generators given by the cycles on the E^2-page of the $RO(C_2)$-graded spectral sequence. Equivalently, $RO(C_2)$-graded bar spectral sequences computing $H_\star K_V$ collapse on the E^2-page.

Proof idea.

Use Hopf ring structure on E^r-page to eliminate nontrivial differentials $\square$

Future directions

Stably,

$$H_\star H \simeq H_\star[\tau_0, \tau_1, \tau_2, \cdots, \xi_1, \xi_2, \cdots]/(\tau_i^2 = (u + a\tau_0)\xi_{i+1} + a\tau_{i+1}).$$

- What does an arbitrary element in $H_\star K_V$ stabilize to in the the C_2-equivariant dual Steenrod algebra?
- How does the stable relation $\tau_i^2 = (u + a\tau_0)\xi_{i+1} + a\tau_{i+1}$ arise unstably?

• Equivariant analogues of Ravenel-Wilson computations

• Algebra of twisted spectral sequences

• $RO(C_2)$-graded bicommutative Hopf rings, C_2-Brown-Gitler Spectra, and Dieudonne theory