# Scissors congruence K theory of manifolds

jt w/ Hoekzema, Murray, Rovi, Semikina

& Raptis, Semikina

<u>Classical question</u>: Given two polyhedra $P, Q$, when are they scissors cong.?

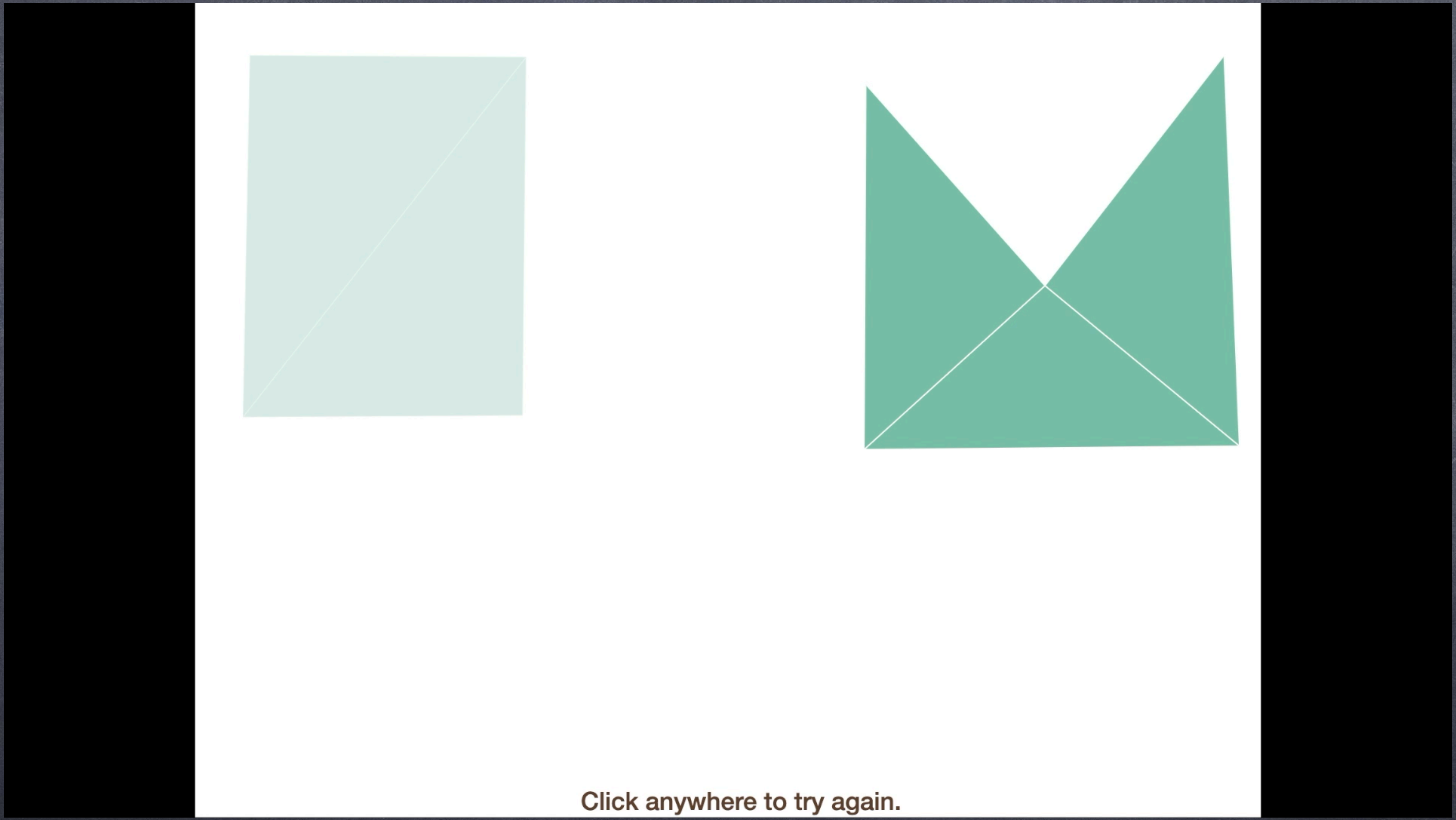
i.e. $P = \bigcup_{i=1}^{n} P_i \qquad Q = \bigcup_{j=1}^{n} Q_j \qquad P_i \cong Q_j$

$P_i$'s $Q_j$'s only intersect in edges & vertices

in 2D: need $P, Q$ to have the same area.

<u>Q</u>: Is this the only SC invariant?

Example: (Wallace - Bolyai - Gerwien theorem)




Animation: Smirnov + Epstein on github

Q: What about dim 3?
(Hilbert's 3rd problem)

Thm: For 3D, the only SC invariants are
- volume
- Dehn invariant $\in \mathbb{R} \otimes \mathbb{R}/\pi\mathbb{Z}$

For arbitrary dim,
open question!

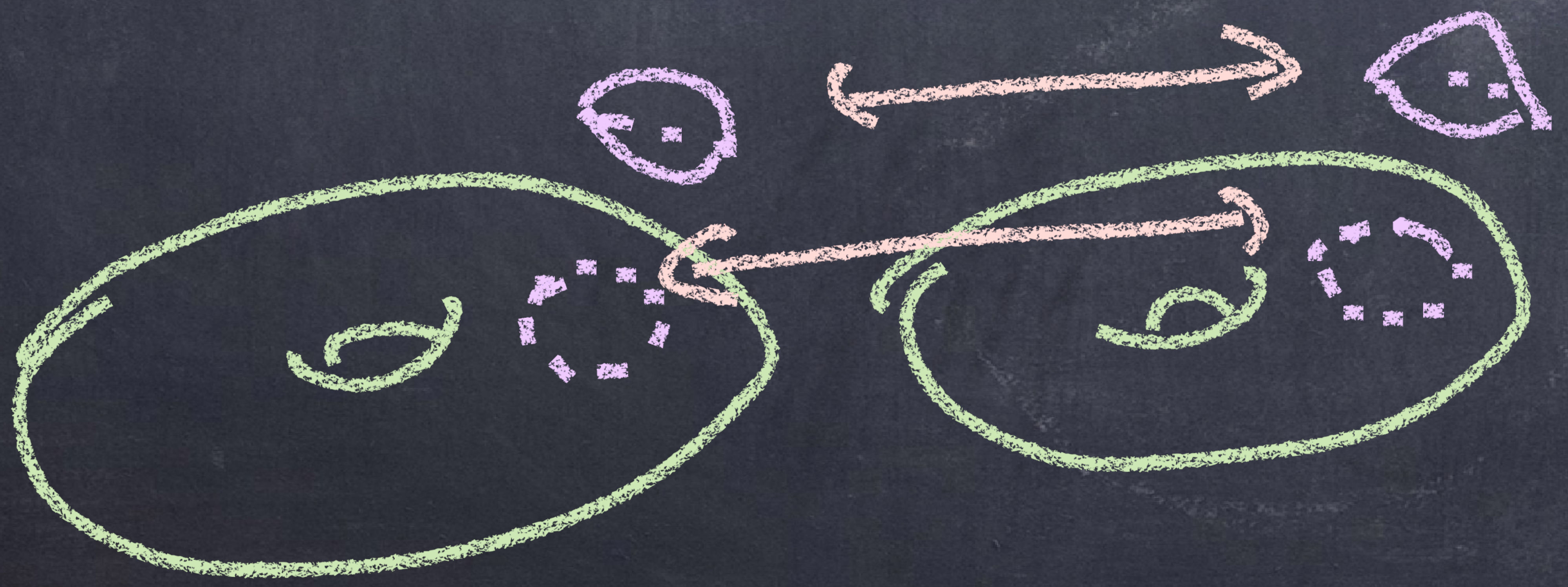
Q: What about dim 3?
(Hilbert's 3rd problem)

Thm (Dehn, Sydler)
In 3D, only SC invariants are
- volume
- Dehn invariant $\in \mathbb{R} \otimes \mathbb{R}/\pi\mathbb{Z}$

For arbitrary dim n:
open question!

# Scissors congruence for compact, oriented smooth manifolds

- start w/ a smooth manifold
- cut along a codim 1 submanifold w/ trivial normal bundle to obtain 2 disjoint pieces
- paste back along a diffeo of the boundaries



SC

SK

**Def:** $N \cup_{\phi} N' \underset{SK}{\simeq} N \cup_{\psi} N'$

"SK equivalent"

SK = "schneiden und Kleben"

= "cut and paste"

(SC "scissors congruent")

**Eg**

<u>Thm:</u> (Karras, Kreck, Neumann, Ossa)
80's "SK book"

The only SK invariants of closed n-dim oriented manifolds are

- $\chi$ Euler char
- $\sigma$ signature

Def: $\mathcal{M}_n$ = monoid of diffeo classes of compact oriented $n$-dim manifolds, $\amalg$

$$SK_n := \frac{Gr(\mathcal{M}_n)}{SK\text{ equiv.}}$$

$\rightsquigarrow$ fully computed in SK gp

# Scissors congruence spectra - brief history

Zakharevich: defines a SC K-theory spectrum via the notion of "assembler"

"Grothendieck site + extra data"

⤳ recovers on $\pi_0$ classical SC gps. for polyhedra + $K_0(Var)$ = free ab gp on var

$$\overline{\quad}$$

$[X] + [Y \setminus X] = [Y]$

Campbell

defines "subtractive cats" and K-theory spectra $\rightsquigarrow$ recovers $K_0(\mathrm{Var})$

$\rightsquigarrow$ their $K(\mathrm{Var})$ agree

Our goal:

can we construct a K-theory spectrum of n-dim manifolds w/ $\pi_0 = SK_n$?

Issue #1:

pieces in a cut & paste need to be objects in your category

$\Rightarrow$ need to work in $\mathcal{M}_n^{\partial}$

Solution to issue #1:

**Def**: for $M, N \in \mathcal{M}_n^\partial$, define cut & paste relation the same way, s.t.

- do not allow boundaries to be cut
- all cut boundaries are required to be pasted back together

$$SK_n^\partial := Gr(\mathcal{M}_n^\partial) / SK \text{ equiv.}$$

**Rmk**: different than the $SK_n^\partial$ proposed in SK-book

$M_1 \cup_\emptyset M_2 \simeq_{SK} M_1 \amalg M_2$ ⚠ we do not allow this

**Thm** (Hoekzema, M, Murray, Rovi, Semikina)

$\exists$ split SES

$$0 \to \underset{[M]}{SK_n} \xrightarrow{\alpha} \underset{[M]}{SK_n^{\partial}} \xrightarrow{\beta} \underset{[\partial M]}{C_{n-1}} \to 0$$

$C_{n-1} \simeq$ Gr (monoid of diffeo classes of null cob (n-1)-dim manifolds)

Pf idea $\boxed{\ker\beta \subseteq \operatorname{im}\alpha}$

$\in \ker\beta$

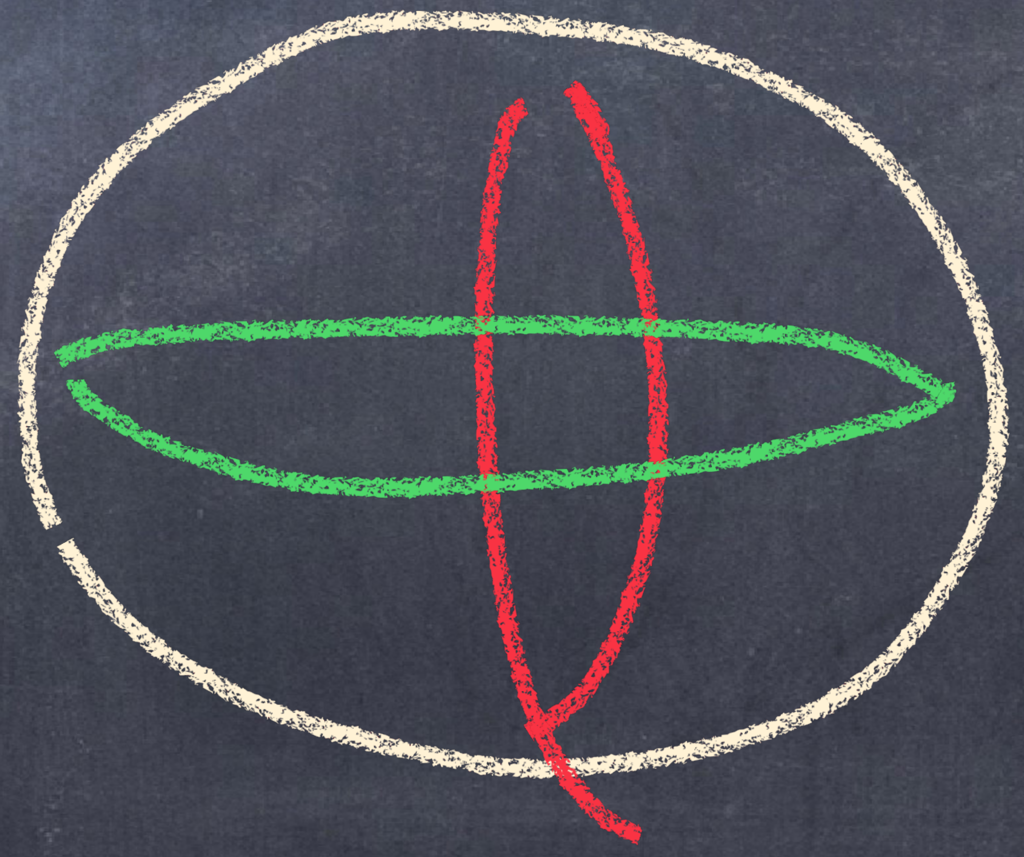
$\parallel$ claim

Pf of claim:

New question: can we recover $SK_n^\partial$ as $\pi_0$ of a K-theory spectrum?

Issue #2: assemblers & subtractive cats do not work

Problem:



covers do not have common subcovers!

# Philosophy behind the problem

- classical K-theory (of exact, Waldhausen, $\infty$-stable cats) "$S_\bullet$-construction"

  encodes relations of the form $[A]+[C]=[B]$
  for exact seq $A \to B \to C$

- assemblers / subtractive cats frameworks

  encode relations of the form $[A]+[C]=[B]$
  for "subtractive seq" $A \to B \leftarrow C$

we really want to encode relations of the form

$[A]+[B]=[C]+[D]$ for

$$\begin{tikzcd} A \arrow[r] \arrow[d] & D \arrow[d] \\ B \arrow[r] & C \end{tikzcd}$$

# K-theory of categories w/ squares

## (Campbell + Zakharevich)

**Def:** A cat $\mathcal{C}$ w/ squares has

- a subcat of "cofibrations" $\rightarrowtail$
- a subcat of "cofiber maps" $\twoheadrightarrow$
- distinguished squares

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & B \arrow[d, two heads] \\ C \arrow[r, tail] & D \end{tikzcd}$$

s.t.

& $0$ initial for $\rightarrowtail$

1) $\mathcal{C}$ has $\amalg$ and $\square$ closed under $\amalg$
2) $\square$ compose horizontally & vertically
3) $\rightarrowtail$, $\twoheadrightarrow$ contain all isos
4) if both $\parallel$ maps in a square are $\cong$, then the sq. is $\square$.

**Def:** $K^{\square}(\mathcal{C}) = \Omega_0 \left| p,q \mapsto \begin{array}{ccccccc} A_{11} & \rightarrowtail & A_{12} & \rightarrowtail & \cdots & \rightarrowtail & A_{1p} \\ \downarrow & \square & \downarrow & & & & \\ A_{21} & \rightarrowtail & \cdot & \cdots & \cdot & & \cdot \\ \vdots & & & & & & \\ \downarrow & \square & & & & \square & \downarrow \\ A_{q1} & \rightarrowtail & \cdot & \cdots & \cdot & & \cdot \end{array} \right|$

**Thm** (Campbell–Zakharevich)

$$K_0^{\square}(\mathcal{C}) = \frac{\text{free ab gp on ob of } \mathcal{C}}{\begin{array}{c}[0]=0 \\ [A]+[D]=[B]+[C] \\ \text{for } \begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow & \square & \downarrow \\ C & \rightarrowtail & D \end{array}\end{array}}$$

& can see: this is an $\infty$ loop space

Def: $\mathrm{Mnfld}^{\partial}_{n}$ is a cat w/ squares w/

- $0 = \emptyset$
- horizontal = vertical maps
  = smooth emb. $M \hookrightarrow N$ s.t. $\partial M$ maps to a submnfld of $N$ w/ triv. normal bundle & each $\partial$ piece maps entirely to the interior or to a $\partial$ piece
- $$\begin{tikzcd} N \arrow[r, tail] \arrow[d, hook] & M \arrow[d, hook] \\ M' \arrow[r, tail] & M \cup_N M' \end{tikzcd}$$ are p.o.

Thm: (HMMRS)

$\mathrm{Mnfld}^{\partial}_{n}$ is a cat w/ squares &

$K_0^{\square}(\mathrm{Mnfld}^{\partial}_{n}) \cong SK^{\partial}_{n}$

Thm

$\exists$ map of spectra $K^{\square}(\mathrm{Mnfld}^{\partial}_{n}) \to K(\mathbb{Z})$

which on $\pi_0$ agrees w/ $\chi$.

Proof idea:

Prop: $\mathcal{C}$ is Waldhausen cat

Then $\mathcal{C}^{\square}$ $\quad \rightarrowtail = \mathrm{cof}$

$\quad \twoheadrightarrow = \underline{\text{all}}$ maps

$$\begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow & & \downarrow \\ C & \rightarrowtail & D \end{array} = \text{sq. w/ } C \cup_A B \xrightarrow{\simeq} D$$

is a cat w/ squares & $K^{\square}(\mathcal{C}) \simeq K^{\mathrm{Wald}}(\mathcal{C})$

(Thomason). $\Gamma$ & reduce to Mayer-Vietoris

Questions:

what are higher htpy gps?

& what higher invariants do they encode?

Def: SKK equivalence (controllable cut & paste)

we keep track of glueing diffeos.

$$[M_1 \cup_\phi M_1'] - [M_1 \cup_\psi M_1'] = [M_2 \cup_\phi M_2'] - [M_2 \cup_\psi M_2']$$

$$= f(\phi, \psi)$$

$$SKK_n = \mathrm{Gr}(\mathcal{M}_n) \big/ \simeq_{SKK} = \pi_1 BCob_n.$$

"Thm" (M.- Roptis - Semikina)

Let $M$ be a $2k+1$-dim, $f$ orientation rev. diffeo.

$$K_1^{\partial}(\text{Mnfld}) \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

$$[M, f] \longmapsto k(M) = \text{Kervaire semichar.}$$

SKK invariant