SYLLABUS FOR ECHT READING SEMINAR ON THE ADAMS SPECTRAL SEQUENCE FOR TOPOLOGICAL MODULAR FORMS

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Prerequisites: Steenrod operations in mod 2 cohomology, spectra and the stable homotopy category, exact couples and spectral sequences.

Omissions: Odd primes, level structures.

(1) **Introduction**
[BR21, Ch. 0].
Quick overview.

(2) **The topological modular forms spectrum**
[Hop95], [Hop02], [Rez], [Bau08], [Goe10], [Kon], [DFHH14, Intro.], [Mat16], [Mil20, Ch. 6].
Sketch the framework leading to the definition of the E_∞ ring spectra

$$tmf \longrightarrow Tmf \longrightarrow TMF\,.$$

Discuss the calculation $H^*(tmf;\mathbb{F}_2) = A//A(2)$, pointing out the role of the Gap Theorem about $\pi_*(Tmf)$.

(3) **The Adams spectral sequence**
[Ada58], [Mos68], [Mos70], [Bru78], [BMMS86, Ch. IV], [Rav86, Ch. 2], [Boa99].
Set up the mod 2 Adams spectral sequence for a spectrum X, with E_2-term $E_2(X) = \mathrm{Ext}_A(H^*(X;\mathbb{F}_2),\mathbb{F}_2) \cong \mathrm{Ext}_{A_*}(\mathbb{F}_2, H_*(X;\mathbb{F}_2))$. Explain the change-of-rings isomorphism $E_2(tmf) = \mathrm{Ext}_{A(2)}(\mathbb{F}_2,\mathbb{F}_2) = \mathrm{Ext}_{A(2)_*}(\mathbb{F}_2,\mathbb{F}_2)$. Discuss detection of homotopy classes by elements in the E_∞-term, conditional and strong convergence, multiplicative structure, and the geometric boundary theorem.

(4) **Minimal resolutions and `ext`**
[BR21, Ch. 1], [BR], [Bru22].
Explain how to use a minimal module resolution to calculate $\mathrm{Ext}_A(-,\mathbb{F}_2)$ or $\mathrm{Ext}_{A(2)}(-,\mathbb{F}_2)$, and lifts of chain maps to calculate the (Yoneda) composition product in Ext. Illustrate using Bruner's program `ext` to establish notation for the algebra generators of $E_2(S)$ in topological degrees $t-s \le 24$ and for $E_2(tmf)$ in degrees $t-s \le 48$. Introduce Steenrod operations in Ext, and report how they act in $E_2(tmf)$.

(5) **The Davis–Mahowald spectral sequence;** Ext **over** $A(2)$
[SI67], [DM82], [BR21, Ch. 2, Ch. 3].
Use the multiplicative Davis–Mahowald spectral sequence to calculate $E_2(tmf) = \mathrm{Ext}_{A(2)_*}(\mathbb{F}_2,\mathbb{F}_2)$ from the groups $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_2, R^\sigma)$, where

$$\mathbb{F}_2 \xrightarrow{\simeq} (A(2)//A(1))_* \otimes R^*$$

is an $A(2)_*$-comodule algebra resolution. Deduce that $\mathrm{Ext}_{A(2)}(\mathbb{F}_2,\mathbb{F}_2) \cong SI$ has the algebra presentation given by Shimada–Iwai. Explain the algorithmic role of a Gröbner basis for this algebra.

(6) **The Adams spectral sequence for** *tmf*
[BR21, Ch. 5, App. A].
Calculate the differential pattern in the mod 2 Adams spectral sequence for *tmf*. Explain (without proof) the formula for $d_*(Sq^i(x))$ and take as known that $\eta^2\kappa = 0$, $\eta\rho \in \{Pc_0\}$ and $\eta^2\bar{\kappa} \in \{Pd_0\}$ in the Adams spectral sequence for S. Summarize presentations of the E_2-, E_3-, E_4- and $E_5 = E_\infty$-terms as R_i-modules, where $R_i = \mathbb{F}_2[w_1, g, w_2^{2^i}]$.

(7) Ext **with coefficients; the Adams spectral sequences for** *tmf*/2 and *tmf*/ν [BR21, Ch. 4, Ch. 6, Ch. 7, Ch. 8].
Use the output of `ext` (in degrees $t-s \leq 72$) to establish notation for R_0-module generators of $E_2(tmf/2) = \mathrm{Ext}_{A(2)}(M_1, \mathbb{F}_2)$ and $E_2(tmf/\nu) = \mathrm{Ext}_{A(2)}(M_4, \mathbb{F}_2)$. Deduce their R_0-module structures from that of $\mathrm{Ext}_{A(2)}(\mathbb{F}_2, \mathbb{F}_2)$. Summarize the differential pattern in the Adams spectral sequences for *tmf*/2 and *tmf*/ν, leading to $E_\infty(tmf/2)$ and $E_\infty(tmf/\nu)$. (The *tmf*/η case is optional.)

(8) **The homotopy groups of** *tmf*
[Bau08], [BR21, Ch. 9.1–9.2].
Introduce names for homotopy classes in $\pi_*(tmf)$ detected by the algebra generators of $E_\infty(tmf)$. Use the results on $E_\infty(tmf/2)$ and $E_\infty(tmf/\nu)$ to determine the hidden 2-, η- and ν-extensions in $E_\infty(tmf)$, and determine $\pi_*(tmf)$ as a graded abelian group with η-, ν-, B- and M-action.

(9) **The algebra structure of** $\pi_*(tmf)$
[DFHH14, Ch. 13], [BR21, Ch. 9.3–9.6].
Discuss the edge homomorphism $\pi_*(tmf) \to mf_{*/2}$ to integral modular forms. Accordingly refine the choices of algebra generators for $\pi_*(tmf)$. Outline how to use the Adams filtration to determine the multiplicative structure of $\pi_*(tmf)$ (up to one sign). Present the alternative algebra generators ($\tilde{B}_k$ in place of B_k) that simplify the multiplicative relations.

(10) **Duality for** *tmf*
[Sto12], [Sto14], [BR21, Ch. 10], [BGR22].
Review Brown–Comenetz and Anderson duality. Express *tmf* as Brown–Comenetz and Anderson duals, and deduce Stojanoska's self-duality of *Tmf*. Introduce the ideal $\Theta\pi_*(tmf) \subset \pi_*(tmf)$. Outline the resulting algebraic dualities in $\pi_*(tmf)$, or the structure of the local cohomology spectral sequence for *tmf*.

(11) H_∞ **ring spectra and Steenrod operations in** Ext
[Mil72], [BMMS86, Ch. I, Ch. VI], [BR21, Ch.11.1–11.2].
Review Steenrod operations in Ext and report how they act in $E_2(S)$ in topological degrees $t-s \leq 24$. Discuss the delayed (= modified) Adams spectral sequence of a tower, and detection of power operations in the homotopy of an H_∞ ring spectrum by Steenrod operations in Ext. Present the formula for the generically first differential $d_*(Sq^i(x))$ on a class x. Deduce the Adams differentials $d_2(h_{i+1}) = h_0h_i^2$.

(12) **The Adams** d**- and** e**-invariants**
[Mau65a], [Mau65b], [Ada66], [BR21, Ch. 11.3], [BR22].
Outline Adams' and Maunder's work on the e-invariant, in terms of factorizations of $S \to ko$ through the fiber of $p\colon ko \to \bigvee_{i>0} \Sigma^{4i} H\mathbb{Z}$, and through the fiber j of $\psi^3 - 1\colon ko \to bspin$. State the (proven) Adams conjecture, and determine the ring structure of $\pi_*(j)$.

(13) **The Adams spectral sequence for** S
[MT67], [BMT70], [Bru84], [BR], [BR21, Ch.11.4–11.7].

Calculate the differential pattern in the mod 2 Adams spectral sequence for S, in the range $t - s \leq 24$. Outline how naturality with respect to $S \to tmf$ helps to determine the differentials in the range $24 < t - s \leq 48$.

(14) **Some homotopy groups of S**
[Isa19], [IWX], [BR21, Ch. 11.8–11.9].
Review names for algebra generators of $\pi_*(S)$ in the range $* \leq 24$. Use the results on $E_\infty(S)$ and the homomorphisms $\pi_*(S) \to \pi_*(j)$ and $\pi_*(S) \to \pi_*(tmf)$ to determine the multiplicative structure of $\pi_*(S)$ in this range. Outline how tmf helps to determine products in the range $24 < t - s \leq 48$.

(15) **The tmf-Hurewicz image**
[BR21, Ch. 11.10–11.11], [BMQ].
Use the cofiber sequence $S \to tmf \to tmf/S$ and the Adams spectral sequence for tmf/S to determine the image of $\pi_*(S) \to \pi_*(tmf)$ in degrees $* \leq 101$. Report on the work by Behrens–Mahowald–Quigley determining the Hurewicz image in all degrees.

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