eCHT Reading Seminar on
Hopf algebras, rings, and algebroids
Organizers: Sarah Petersen, TBD

1. Introduction

Hopf algebras arise naturally from the homology of spaces with multiplications (i.e. H-spaces, or "Hopf" spaces). In the language of category theory, the homology of such spaces is a group object in the category of coalgebras. When spaces have additional structure, this is reflected in their homology. For example, the spaces classifying generalized homology theories have a structure mimicking that of a graded ring. In turn, their homology has the structure of a Hopf ring, which is a ring in the category of coalgebras. Generalized homology theories lead to a generalization of the definition of a Hopf algebra to that of a Hopf algebroid, a cogroupoid object in the category of commutative algebras over a commutative ground ring K. The goal of this seminar is to study how Hopf algebras, rings, and algebroids lead to new computational techniques as well as elegant descriptions of answers in algebraic topology.

2. Schedule

We will meet for an hour on Tuesdays at 11 am eastern time. There will also be office hours on Thursdays from noon to 1 pm eastern time.

2.1. **January 16, Overview.** (Petersen) This talk will outline the topics covered in the rest of the seminar.

2.2. **January 23, The Steenrod Algebra and Its Dual.** Briefly summarize the role of the Steenrod algebra, or equivalently its dual, in Adams spectral sequence computations. Describe Milnor's work showing that the Steenrod algebra and its dual are in fact Hopf algebras.

Suggested reading: Milnor. *The Steenrod Algebra and Its Dual* [9]

2.3. **January 30, The homology of Eilenberg-MacLane spaces.** Introduce Hopf rings and Hopf rings in the bar spectral sequence [16, §7]. Compute the mod p-homology of mod p-Eilenberg-MacLane spaces following [16, §8]

Suggested reading: [16, §7 and §8] - Wilson. *Brown-Peterson Homology: An Introduction and Sampler*

2.4. **February 6, The Hopf ring for complex cobordism.** Review the structure of Hopf rings in [13, §1]. Describe the spaces of the spectrum MU (see [13, §2]). Deduce the main formal group law relations [13, §3]. State and outline proof of main theorem [13, §4]

Suggested reading: Ravenel–Wilson *The Hopf ring for complex cobordism* [13]

2.5. **February 13, The structure of spaces representing a Landweber exact cohomology theory.** State [4, Theorem 2.2] and give a sketch of the proof.

Suggested reading: Hopkins–Hunton. *On the structure of spaces representing a landweber exact cohomology theory* [4]

2.6. **February 20, Hopf rings, Dieudonné modules, and $E_*\Omega^2S^3$.** Introduce Dieudonné theory [2, §4]. Discuss the Hopf ring for complex cohomoloy theories [2, §10]. If time permits, introduce the role of $E_*(\Omega^2S^3)$.

Suggested reading: Goerss. *Hopf Rings, Diudonné Modules, and* $E_*\Omega^2S^3$ [2]

2.7. **February 27, The connective real K-theory of Brown–Gitler spectra.** Recall the Dieudonné ring and Hopf rings for the mod 2 Eilenberg-MacLane spectrum [10, §6]. Calculate the $s = 0$ line of the Adams spectral sequence for ko-homology [10, §7]. Overview calculation of the E_2-term of the Adams spectral sequence for the ko-homology of $B(2n)$ up to stable isomorphism, thereby determining the unstable classes in the Dieudonné ring for ko [10, §8].

Suggested reading: Pearson. *The connective real K-theory of Brown–Gitler spectra* [10]

2.8. **March 5, The mod-two cohomology rings of symmetric groups.** Introduce the Hopf ring in cohomology [1, §2]. State and overview proof of [1, Theorem 1.2].

Suggested reading: Giusti–Salvatore–Sinha. *The mod-two cohomology rings of symmetric groups* [1]

2.9. **March 12, The $RO(C_2)$-graded homology of C_2-equivariant Eilenberg-MacLane spaces.** Introduce $RO(C_2)$-graded homology theories and C_2-equivariant Eilenberg-MacLane spaces [11, §3], [8]. Describe the C_2-equivariant bar and twisted bar constructions [11, §4]. Describe the multiplicative structures on C_2-equivariant Eilenberg-MacLane spaces [11, §5]. Overview the bar and twisted bar spectral sequence computations [11, §6].

Suggested reading: Petersen. The $H\underline{\mathbb{F}}_2$-homology of C_2-equivariant Eilenberg-MacLane spaces [11], Lewis *Equivariant Eilenberg-Mac Lane spaces and the equivariant Seifert-van Kampen and suspension theorems* [8]

2.10. **March 19, Real Wilson spaces.** Describe the universal Hopf ring approach of [3, §1]. Develop sufficient background to state the main theorem [3, Proposition 5.6]

Suggested reading: Hill–Hopkins. Real Wilson Spaces I [3]

2.11. **March 26, Break.** *No seminar this week.*

2.12. **April 2, Hopf algebroid and homological algebra.** Define a Hopf algebroid. Describe the change of rings isomorphisms [12, A1.1.18]. Cover some of the homological algebra of comodules over Hopf algebroids. Remark on spectral sequences useful for computing Ext over a Hopf algebroid. Reference: [12, Appendix A1].

Suggested reading: Ravenel. *Complex cobordism and stable homotopy groups of spheres* (also known as the Green Book) Appendix A1 [12]

2.13. **April 9, The C_p-equivariant dual Steenrod algebra.** Overview, compare, and contrast the computations and results of [5, 14], and [6].

Suggested reading: Hu–Kriz *Real-oriented homotopy theory and an analogue of the AdamsNovikov spectral sequence* [5, §6]. Sankar–Wilson *On the C_p-equivariant dual Steenrod algebra* [14]. Hu–Kriz–Somberg–Zou *The $\mathbb{Z}/p$-equivariant dual Steenrod algebra for an odd prime p* [6].

2.14. **April 16, The Hopf ring for $ER(n)$.** State [7, Theorem 1.1] and give an overview of the proof.

Suggested reading: Kitchloo–S. Wilson *On the Hopf Ring for $ER(n)$* [7]

2.15. **April 23, Motivic Twisted K-theory.** State [15, Theorem 2.2], a motivic analogue of the universal coefficient theorem for twisted K-theory, and give an overview of the proof.

Suggested reading: Spitzweck–Østvær. *Motivic Twisted K-theory* [15]

2.16. **April 30, TBA.**

References

[1] Chad Giusti, Paolo Salvatore, and Dev Sinha, *The mod-2 cohomology rings of symmetric groups*, J. Topol. **5** (2012), no. 1, 169–198. MR2897052

[2] Paul G. Goerss, *Hopf rings, Dieudonné modules, and $E_*\Omega^2S^3$*, Homotopy invariant algebraic structures (Baltimore, MD, 1998), 1999, pp. 115–174. MR1718079

[3] Michael A. Hill and Michael J. Hopkins, *Real wilson spaces i*, 2018.

[4] M. J. Hopkins and J. R. Hunton, *On the structure of spaces representing a Landweber exact cohomology theory*, Topology **34** (1995), no. 1, 29–36. MR1308488

[5] Po Hu and Igor Kriz, *Real-oriented homotopy theory and an analogue of the Adams-Novikov spectral sequence*, Topology **40** (2001), no. 2, 317–399. MR1808224

[6] Po Hu, Igor Kriz, Petr Somberg, and Foling Zou, *The $\mathbb{Z}/p$-equivariant dual steenrod algebra for an odd prime p*, 2023.

[7] Nitu Kitchloo and W. Stephen Wilson, *On the Hopf ring for $ER(n)$*, Topology Appl. **154** (2007), no. 8, 1608–1640. MR2317067

[8] L. Gaunce Lewis Jr., *Equivariant Eilenberg-Mac Lane spaces and the equivariant Seifert-van Kampen and suspension theorems*, Topology Appl. **48** (1992), no. 1, 25–61. MR1195124

[9] John Milnor, *The Steenrod algebra and its dual*, Ann. of Math. (2) **67** (1958), 150–171. MR99653

[10] Paul Thomas Pearson, *The connective real K-theory of Brown-Gitler spectra*, Algebr. Geom. Topol. **14** (2014), no. 1, 597–625. MR3158769

[11] Sarah Petersen, *The $h\underline{\mho}_2$-homology of c_2-equivariant eilenberg-maclane spaces*, 2022.

[12] Douglas C. Ravenel, *Complex cobordism and stable homotopy groups of spheres*, Pure and Applied Mathematics, vol. 121, Academic Press, Inc., Orlando, FL, 1986. MR860042

[13] Douglas C. Ravenel and W. Stephen Wilson, *The Hopf ring for complex cobordism*, J. Pure Appl. Algebra **9** (1976/77), no. 3, 241–280. MR448337

[14] Krishanu Sankar and Dylan Wilson, *On the c_p-equivariant dual steenrod algebra*, 2021.

[15] Markus Spitzweck and Paul Arne Ø stvær, *Motivic twisted K-theory*, Algebr. Geom. Topol. **12** (2012), no. 1, 565–599. MR2916287

[16] W. Stephen Wilson, *Brown-Peterson homology: an introduction and sampler*, CBMS Regional Conference Series in Mathematics, vol. 48, Conference Board of the Mathematical Sciences, Washington, DC, 1982. MR655040