Suggested Exercises for
eCHT Research Workshop on Hopf Rings
June 24 - 28, 2024
Organizers: Sarah Petersen

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Suggested Use. This document contains a wide sampling of introductory exercises related to Hopf algebras, rings, and algebroid in algebraic topology. It is recommended to skim the entire article, and then select a smaller subset of exercises that interest you to work through before the workshop begins.

1. Hopf Algebras

These definitions and exercises are meant to accompany Milnor's The Steenrod algebra and its dual [8].

Definition 1.1. A bialgebra is a monoid in the category of coalgebras or equivalently a comonoid in the category of algebras.

Definition 1.2. A Hopf algebra is an (associative and coassociative) bialgebra H over a field k together with a k-linear map $S : H \to H$ (called the antipode) such that the diagram

$$\begin{tikzcd}
 & H \otimes H \arrow[rr, "S \otimes id"] \arrow[dl, "\Delta"'] & & H \otimes H \arrow[dr, "\nabla"] & \\
H \arrow[rr] \arrow[dr, "\Delta"'] & & \mathrm{k} \arrow[rr] & & H \arrow[dl, "\nabla"] \\
 & H \otimes H \arrow[rr] & & H \otimes H &
\end{tikzcd}$$

commutes. Here Δ is the comultiplication of the bialgebra, ∇ is the multiplicaiton,η is the unit, and η is the counit.

1.1. **Hopf algebra exercises.**

(1) This problem will lead you to prove that Milnor's definiton of a connected Hopf algebra (A_*, ψ_*, ϕ_*) as a connected graded algebra with unit (A_*, ψ_*) together with a homomorphism

$$\phi_* : A_* \to A_* \otimes A_*$$

satisfying usual properties of a graded coproduct [8, Properties 2.1 & 2.3] is a graded instance of the above definition. We will make use of the following definition.

Definition 1.3. [9, Definition 8.1] If A is a connected coalgebra and B is a connected algebra, let $G(A, B)$ denote the set of morphisms of modules $f : A \to B$ such that f_0 is the identity on k. If $f, g \in G(A, B)$, let $f * g$ be the composition

$$A \xrightarrow{\Delta} A \otimes A \xrightarrow{f \otimes g} B \otimes B \xrightarrow{\varphi} B.$$

(a) Prove [9, Proposition 8.2] If A is a connected coalgebra, then $G(A, B)$ is a group under the operation $*$ with identity

$$A \xrightarrow{\epsilon} k \xrightarrow{\eta} B.$$

(b) If A is a connected Hopf algebra, use part (a) to define the antipode on A [9, Definition 8.4].

(c) Show Milnor's definition of a connected Hopf algebra $(A_*, \psi_*, \phi_*$ satisfies the properties of Definition 1.2.

2. Ravenel–Wilson Hopf ring techniques

The exercises in this section are meant to give a solid introduction to Ravenel–Wilson style Hopf ring techniques. They accompany the results of [19, 14, 4, 12, 3].

2.1. The mod p-homology of Eilenberg–MacLane spaces exercises.

(1) Show that the tensor product is a product in the category of coalgebras.

(2) Show that a Hopf algebra is a group object in the category of coalgebras.

(3) Show that a Hopf ring is a ring in the category of coalgebras and deduce the associativity law.

(4) Let

$$E(x) = \mathbb{Z}/p\mathbb{Z}\,[x]/(x^2) \qquad \text{and} \qquad T(x) = \mathbb{Z}/p\mathbb{Z}\,[x]/(x^p)$$

denote the exterior algebra and height p truncated polynomial algebra, respectively. Compute:

(a) $\mathrm{Tor}^{E(x)}(\mathbb{F}_p, \mathbb{F}_p)$

(b) $\mathrm{Tor}^{T(x)}(\mathbb{F}_p, \mathbb{F}_p)$ (it may be helpful to see [10, Borel transgression Theorem 12.11.2])

(5) Write the Serre generators for the cohomology of the first few Eilenberg–MacLane spaces (see, for instance, the section on the cohomology of Eilenberg-MacLane spaces in Hatcher's Spectral Sequences Chapter). Compare these with the Ravenel–Wilson homology generators for the same spaces.

(6) Use the bar spectral sequence and bar spectral sequence pairing to compute

$$E^2_{*,*} \simeq Tor^{H_*(K(\mathbb{Z}/p\mathbb{Z},0);\mathbb{F}_p)}(\mathbb{F}_p, \mathbb{F}_p) \implies H_*K_1 \cong E[e_1, \alpha_{(i)}]$$

and

$$E^2_{*,*} \simeq Tor^{H_*(K(\mathbb{Z}/p\mathbb{Z},1);\mathbb{F}_p)}(\mathbb{F}_p, \mathbb{F}_p) \implies H_*K_2 \cong E[e_1 \circ \alpha_{(i)}, \alpha_{(i_1)} \circ \alpha_{(i_2)}, \beta_{(j)}]$$

in the notation from pg. 52 of the Brown-Peterson homology sampler.

2.2. The Hopf ring for complex cobordism exercises.

(1) Let $\mathcal{C}$ be a category with finite products and a terminal object $N_{\mathcal{C}}$, $\mathcal{D}$ be a category with finite products and a terminal object $N_{\mathcal{D}}$, and $F : \mathcal{C} \to \mathcal{D}$ be a product preserving functor, that is, a functor $\mathcal{F}$ such that $\mathcal{F}(N_{\mathcal{C}}) = N_{\mathcal{D}}$ and there is a natural equivalence of functors

$$\mathcal{F}(-)\Pi\mathcal{F}(-) \simeq \mathcal{F}(-\Pi-)$$

from $\mathcal{C} \times \mathcal{C}$ to $\mathcal{D}$. Prove that if X_* is a graded ring object over $\mathcal{C}$, then $\mathcal{F}(X_*)$ is a graded ring object over $\mathcal{D}$ [14, Lemma 1.1].

(2) The cobordism group of complex maps of codimension $2n$ is the complex bordism group MU_*MU_n, and MU_*MU is the cobordism group of all maps with even codimension [18]. The additive and multiplicative products in MU induce products in MU_*MU_* which can be described geometrically. First observe that the additive product is the H-space structure on MU, which arises from the fact that it is a loop space; the multiplicative product is induced by the Whitney sum maps

$$MU_n \wedge MU_m \to MU_{n+m}.$$

Prove that if $f_i : M_i \to N_i$, $i = 1, 2$ represents an element of MU_*MU, then their multiplicative and additive products are represented by

$$f_1 \times f_2 : M_1 \times M_2 \to N_1 \times N_2$$

and

$$f_1 \times 1 \sqcup 1 \times f_2 : M_1 \times N_2 \sqcup N_1 \times M_2 \to N_1 \times N_2$$

respectively [14, Proposition 2.2].

(3) Recall $E^*CP^\infty \cong E^*[[x]]$. Let $\beta_n \in E_{2n}CP^\infty$ be dual to x^n and define

$$\beta(r) = \sum_{i \geq 0} \beta_i r^i.$$

Unpack the notation in the proof of [14, Theorem 3.2] to show

$$\beta(s)\beta(t) = \beta(s +_F t).$$

2.3. **The structure of spaces representing a Landweber exact cohomology theory exercises.**

(1) Let $E = \{\mathbf{E}_r\}$ be a multiplicative Ω-spectrum with a Künneth isomorphism and consider $v \in \pi_{-r}(E)$ as an unbased map

$$v : \text{ point } \to \mathbf{E}_r.$$

Select a generator γ of $H_0(\text{point}, R)$, where R is a commutative ring. Define $[v]$ as the image under v_* of γ. Show

$$\begin{aligned} [v] * [w] &= [v + w] \\ [v] \circ [w] &= [vw] \\ \psi[v] &= [v] \otimes [v]. \end{aligned}$$

(2) Let $E^*(-)$ be a complex oriented cohomology theory. Show that there is a ring map $MU_* \to E_*$ (this is either trivial or a theorem depending on your definition of complex oriented). Furthermore, observe that if

$$MU_*(-) \otimes_{MU_*} E_*$$

is a homology theory, then this homology theory is naturally isomorphic to $E_*(-)$ on finite complexes.

(3) Landweber's original statement of the exact functor theorem says that a ring map $MU_* \to E_*$ specifies a homology theory

$$MU_*(-) \otimes_{MU_*} E_*$$

if and only if, for each prime p, the image of the sequence $v_0, v_1, \ldots$ in E_* is regular. Note that $H_*(-;\mathbb{Z}/p)$ and $KU_*(-)$ are both complex oriented, so that we have ring maps

$MU_* \to \mathbb{Z}/p$ and $MU_* \to KU_*$. Using the LEFT, show that the first map does not define a homology theory, but that the second map does.

Hint: Use the fact that the formal group law corresponding to $H^*(-;\mathbb{Z}/p)$ is

$$F(x,y) = x + y$$

and the formal group law corresponding to $KU^*(-)$ is

$$F(x,y) = x + y - v_1 xy$$

2.4. **The $H\underline{\mathbb{F}}_2$-homology of C_2-equivariant Eilenberg–MacLane spaces exercises.**

(1) Convince yourself that $K(\mathbb{F}_2, \sigma) \simeq B^\sigma \mathbb{F}_2$ is the space of lines in $\mathbb{R}^{\infty\rho}$.

(2) Try this exercise if you haven't seen equivariant cohomology theories before.

Classically, the best way to compute things like $H\mathbb{F}_2^*(X)$ is to have a cell structure on X and use the cellular cochains. We can do a similar thing equivariantly. Equivariantly we have G-cells which are spaces of the form $G/H_+ \wedge D^n$ for subgroups $H \subset G$ and $n \in \mathbb{N}$. If we can construct a G-space X inductively by attatching n-cells to a X^{n-1} via equivariant maps

$$G/H_+ \wedge \partial D^n \to X^{n-1}$$

we call X a G-CW complex. Suppose X has a G-equivariant cell structure and let $C_n(X)$ be the free abelian group on the set of n-dimensional G-cells. Note that the generators of this group have a natural G-action. Then for an abelian group A with *trivial* C_2-action we can consider equivariant homomorphisms

$$\mathrm{Hom}_{\mathbb{Z}}^G(C_n(X), A)$$

and nonequivariant homomorphisms

$$\mathrm{Hom}_{\mathbb{Z}}(C_n(X), A).$$

Each will give rise to a chain complex in the usual way. The first of these computes the top level of the cohomology Mackey functor $H\underline{A}_G^\star(X)$ and the latter computes the underlying. Note: the coefficients here are the constant Mackey functor $\underline{A}$.

(a) Give C_2-equivariant cell structures to $S^\sigma, S^{2\sigma}$, and $S^{1+\sigma}$

(b) Compute the cohomology C_2-Mackey functors $(H\mathbb{F}_2)^\star_{C_2}(X)$ for each of the X above using the G-cell structures.

(3) Show that for $a : S^0 \to S^{-\sigma}$ the inclusion of fixed points, the cofiber is $Ca \simeq (C_2)_+$ in Sp^G.

(4) Check that

$$(H\mathbb{F}_2)^{C_2}_\star(\mathbb{F}_2^{\wedge t}) \cong (H\mathbb{F}_2)^{C_2}_\star(\mathbb{F}_2)^{\otimes t},$$

where $\mathbb{F}_2^{\wedge t}$ has the C_2-action

$$(a_1, ..., a_t) \mapsto (a_t, ..., a_1).$$

[Hint: This is a norm. Can use this.]

(5) Try and work out the Bar and twisted Bar spectral sequences computing $(H\mathbb{F}_2)^{C_2}_\star(K_{1+\sigma})$.

(6) Prove that the diagonal map $\Delta : K_V \to K_V \wedge K_V$ gives rise to a coproduct

$$(H\mathbb{F}_2)^{C_2}_\star(K_V) \to (H\mathbb{F}_2)^{C_2}_\star(K_V) \otimes_{(H\mathbb{F}_2)} (H\mathbb{F}_2)^{C_2}_\star(K_V).$$

Compute the coproduct on the generators $(e_\sigma \bar{\alpha}_{(n)})^J$ of $(H\mathbb{F}_2)^{C_2}_\star(K_{*\sigma})$.

3. Hopf rings and Dieudonné theory

The exercises in this section are meant to serve as an introduction to Hopf rings and Dieudonné theory. They accompany the results of [2, 11].

3.1. **Hopf Rings, Dieudonné Modules, and $E_*\Omega^2 S^3$ exercises.**

(1) Throughout we use the conventions that $x = (x_0, x_1, \cdots x_n)$ and $\mathbb{Z}_p$ denotes the p-adic integers. Let

$$w_n = w_n(x) = w_n(x_0, x_1, \cdots, x_n) = x_0^{p^n} + p x_1^{p^{n-1}} + \cdots + p^n x_n$$

be the nth Witt polynomial. Use the Dwork Lemma [2, Lemma 1.1] to deduce there exists a unique polynomial

$$a_i(x, y) \in \mathbb{Z}_p[x_0, x_1, \cdots, y_0, y_1]$$

such that

$$w_n(a_0, a_1, \cdots) = w_n(x) + w_n(y)$$

.

(2) Use the result of the previous exercise to define a coproduct

$$\Delta : \mathbb{Z}_p[x_0, x_1, \cdots, x_n] \to \mathbb{Z}_p[x_0, x_1, \cdots, x_n] \otimes \mathbb{Z}_p[x_0, x_1, \cdots, x_n]$$

by

$$\Delta(x_i) = a_i(x \otimes 1, 1 \otimes x).$$

Show that this coproduct makes $\mathbb{Z}_p[x_0, x_1, \cdots, x_n]$ a bicommutative Hopf algebra over $\mathbb{Z}_p$. Show also that the Witt polynoimals $w_n(x)$ are primative [2, Lemma 1.4].

(3) Read [2, Remark 1.5]

(4) Work through the proof of and definitions involved in [2, Proposition 2.2].

(5) Work through the statement of [2, Lemma 2.5] as well as the two preceding sentences.

(6) Read [2, Lemma 1.5].

(7) Calculate the Dieudonné-module for $H_*(BU, \mathbb{F}_2)$ following [2, Example 4.12]. You will need to use [2, Theorem 4.8].

(8) See [2, §10], and in particular, Example 10.4, for a Dieudonné perspective on the Hopf ring of complex oriented cohomology theories (note: much of the exposition is fairly technical, but one can see the role of the formal group laws we've been studying the past few weeks fairly directly).

3.2. **The connective real K-theory of Brown–Gitler spectra exercises.**

(1) Work through the proof that the destabilization function $e^{-\infty} : H_*H\mathbb{F} \to H_*\underline{H\mathbb{F}}_*$ is well-defined (Lemma 6.6).

(2) Work through the proof that the destabilization function $e^{-\infty} : H_*E \to H_*E \otimes H_*B(\infty) \otimes \Lambda$ of Definition 7.1 is well-defined (Lemma 7.2).

(3) In Section 8, Pearson uses induction to compute $ko_*B(2n)$. There are four cases: $2n \equiv 0, 2, 4, 6 \bmod 8$. Try working out any of the cases on your own. Hint: $2n \equiv 0, 4$ are slightly simpler, so I'd suggest starting there.

4. Hopf rings in cohomology

The exercises in this section are meant to introduce Hopf rings in cohomology. They accompany the results of [1].

4.1. The mod-2 cohomology rings of symmetric groups exercises.

(1) Consider the Hopf ring given by the total symmetric invariants $\boldsymbol{k}[x]^S$ [1, see Definition 2.1, Proposition 2.3, and Example 2.6].
   (a) Compute the coproduct
   $$\Delta_{2,1}(x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2)$$
   (b) and the product
   $$(x_1^2x_2 + x_1x_2^2) \odot x_1$$
   .
   (c) Read the rest of [1, Example 2.6]
(2) Unpack and generalize the "linear" geometric representatives for homology classes appearing in [1, Figures 1 and 2] on pages 12 and 13 (in the arXiv version).

5. Hopf algebroids and computations

These exercises are meant to give an introduction to working with Hopf algebroids. They accompany [13, Appendix A1] and [5].

5.1. Hopf algebroid exercises.

These exercises are meant to accompany [13, Appendix A1].

(1) Show $\mathbb{G}_m(A) = A^\times$ is corepresented by $k[x^\pm]$. Use the group structure of $A^\times$ to equip $k[x^\pm]$ with a Hopf algebra structure.
(2) Suppose E_*E is flat as a left $\pi_*(E)$-module. Check that the
   (a) comultiplication
   $$E_*E \to \pi_*(E \wedge E \wedge E) \cong E_*E \otimes_{\pi_*E} E * E,$$
   (b) counit
   $$\epsilon : E_*E \to \pi_*E$$
   induced by the map $E \wedge E \to E$,
   (c) left and right unites η_L and η_R corresponding to the left and right module structures on E_*E corresponding to the left and right module structures on E_*E, and
   (d) antipode $S : E_*E \to E_*E$ induced by the swap map $S : E \wedge E \to E \wedge E$
   satisfy the diagrams defining a Hopf algebroid.

5.2. The C_2-equivariant dual Steenrod algebra exercises.

(1) Show that the cofiber S^0/a^n is equivalent to $\Sigma^{1-n\sigma}S^{n-1}_{\text{anti}\,+}$.
(2) Verify that $F(\Sigma^{n\sigma-1}S^0/a^n, X)$ is equivalent to X/a^n.
(3) Use the relations
   $$\zeta_1 = a\xi_1 + \tau_0 \qquad \text{and} \qquad a^2\tau_1 = u\tau_0 + u\zeta_1 + a\zeta_1\tau_0$$
   in $\pi_\star F(EC_{2+}, H \wedge H)$ to verify that $\tau_0^2 = u\xi_1 + a\tau_0\xi_1 + a\tau_1$.

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