Suggested Reading and References for
eCHT Research Workshop on Hopf Rings and Computations in Homotopy Theory
June 24 - 28, 2024
Organizers: Sarah Petersen

1. Introduction

Hopf algebras arise naturally from the homology of spaces with multiplications (i.e. H-spaces, or "Hopf" spaces). In the language of category theory, the homology of such spaces is a group object in the category of coalgebras. When spaces have additional structure, this is reflected in their homology. For example, the spaces classifying generalized homology theories have a structure mimicking that of a graded ring. In turn, their homology has the structure of a Hopf ring, which is a ring in the category of coalgebras. Generalized homology theories lead to a generalization of the definition of a Hopf algebra to that of a Hopf algebroid, a cogroupoid object in the category of commutative algebras over a commutative ground ring K. The goal of this research workshop is to study how Hopf algebras, rings, and algebroids lead to new computational techniques as well as elegant descriptions of answers in algebraic topology.

2. Schedule

We will meet from 11 AM - 2 PM Eastern Time on June 24-28.

3. Suggested Reading & References

3.1. **Hopf algebras.** Milnor's The Steenrod algebra and its dual is an excellent first place to read about Hopf algebras.

[15] John Milnor. "The Steenrod algebra and its dual". In: *Ann. of Math. (2)* 67 (1958), pp. 150–171. ISSN: 0003-486X. DOI: `10.2307/1969932`. URL: `https://doi.org/10.2307/1969932`.

3.2. **Ravenel–Wilson Hopf ring techniques.** Sections 7 and 8 of Wilson's Brown–Peterson homology: an introduction and sampler provide an excellent first introduction to Ravenel–Wilson Hopf ring techniques [26]. In particular, a Hopf ring structure in the bar spectral sequence is introduced and the mod p-homology of mod p-Eilenberg-MacLane spaces is computed. We note that while the arguments are elegant, the notation required is at times densely technical and thus may require careful attention from the reader.

[26] W. Stephen Wilson. *Brown-Peterson homology: an introduction and sampler*. Vol. 48. CBMS Regional Conference Series in Mathematics. Conference Board of the Mathematical Sciences, Washington, DC, 1982, pp. v+86. ISBN: 0-8219-1699-3

From there, Ravenel and Wilson's computations of the Hopf ring for complex cobordism [21] and the Morava K-theories of Eilenberg-MacLane spaces and the Conner-Floyd conjecture [22] are examples where the techniques of [26] are applied in broader context.

[21] Douglas C. Ravenel and W. Stephen Wilson. "The Hopf ring for complex cobordism". In: *J. Pure Appl. Algebra* 9.3 (1976/77), pp. 241–280. ISSN: 0022-4049,1873-1376. DOI: `10.1016/0022-4049(77)90070-6`. URL: `https://doi.org/10.1016/0022-4049(77)90070-6`

[22] Douglas C. Ravenel and W. Stephen Wilson. "The Morava K-theories of Eilenberg-Mac Lane spaces and the Conner-Floyd conjecture". In: *Amer. J. Math.* 102.4 (1980), pp. 691–748. ISSN: 0002-9327,1080-6377. DOI: 10.2307/2374093. URL: https://doi.org/10.2307/2374093

Due to the fact that the complex bordism spectrum MU is the universal complex-oriented cohomology theory, the computation of the Hopf ring for complex cobordism also led to computations by Hopkins–Hunton and Hunton–Turner on the structure of spaces representing a Landweber exact cohomology theory [9], related coalgebraic algebra [12], and a calculation of the homology of spaces representing exact pairs of homotopy fucntors [13].

[9] M. J. Hopkins and J. R. Hunton. "On the structure of spaces representing a Landweber exact cohomology theory". In: *Topology* 34.1 (1995), pp. 29–36. ISSN: 0040-9383. DOI: 10.1016/0040-9383(94)E0013-A. URL: https://doi.org/10.1016/0040-9383(94)E0013-A

[12] John R. Hunton and Paul R. Turner. "Coalgebraic algebra". In: *J. Pure Appl. Algebra* 129.3 (1998), pp. 297–313. ISSN: 0022-4049,1873-1376. DOI: 10.1016/S0022-4049(97)00076-5. URL: https://doi.org/10.1016/S0022-4049(97)00076-5

[13] John R. Hunton and Paul R. Turner. "The homology of spaces representing exact pairs of homotopy functors". In: *Topology* 38.3 (1999), pp. 621–634. ISSN: 0040-9383. DOI: 10.1016/S0040-9383(98)00037-8. URL: https://doi.org/10.1016/S0040-9383(98)00037-8

More recently, Ravenel–Wilson style Hopf ring techniques have begun to be studied in equivariant and motivic homotopy theory. In particular, Hill and Hopkins have a preprint [8], which refines a universal Ravenel–Wilson property and studies some C_2-equivariant questions similar to those appearing in Ravenel and Wilson's The Hopf ring for complex cobordism. Additionally, Petersen extended the Ravenel–Wilson Hopf ring techniques of [26, Sections 7 and 8] to compute the $RO(C_2)$-graded homology of C_2-equivariant Eilenberg–MacLane spaces [19]. Finally, Hopkins and Bachmann have work which computes the motivic cohomology of motivic Eilenberg–MacLane spaces, also using Ravenel–Wilson style Hopf ring techniques.

[8] Michael A. Hill and Michael J. Hopkins. *Real Wilson Spaces I.* 2018. arXiv: 1806.11033 [math.AT]

[19] Sarah Petersen. *The $H\underline{\mathbb{F}}_2$-homology of C_2-equivariant Eilenberg-MacLane spaces.* 2022. arXiv: 2206.08165 [math.AT]

[1] Tom Bachmann and Michael J. Hopkins. *Cohomology of motivic Eilenberg–MacLane spaces.* (preprint not yet available). 2022

3.3. **Hopf rings and Dieudonné theory.** In addition to Ravenel–Wilson Hopf ring techniques, there is another perspective due to Paul Goerss that recovers many of the nonequivariant computations above [6]. In particular, Goerss built on the work of Schoeller, who showed that the abelian category of bicommutative Hopf algebras over the prime field with p elements is equivalent to a category of graded modules, known as Dieudonné modules, to point out that Hopf rings can be studied by looking at an associated ring object in Dieudonné modules.

[6] Paul G. Goerss. "Hopf rings, Dieudonné modules, and $E_*\Omega^2S^3$". In: *Homotopy invariant algebraic structures (Baltimore, MD, 1998).* Vol. 239. Contemp. Math. Amer. Math. Soc., Providence, RI, 1999, pp. 115–174. ISBN: 0-8218-1057-X. DOI: 10.1090/conm/239/03600. URL: https://doi.org/10.1090/conm/239/03600

Paul Pearson implemented Goerss' Dieudonné module perspective in a number of computational examples. Specifically, he used Hopf algebra and Dieudonné module techniques to compute the mod 2-homology of the connective spectrum of topological modular forms, the connective real K-theory of Brown–Gitler spectra, and the mod 2 Hopf ring for connective Morava K-theory.

[17] Paul Thomas Pearson. *The mod 2 homology of the connective spectrum of topological modular forms.* Thesis (Ph.D.)–Northwestern University. ProQuest LLC, Ann Arbor, MI, 2006, p. 155. ISBN: 978-0542-62139-0

[16] Paul Thomas Pearson. "The connective real K-theory of Brown-Gitler spectra". In: *Algebr. Geom. Topol.* 14.1 (2014), pp. 597–625. ISSN: 1472-2747,1472-2739. DOI: 10.2140/agt.2014.14.597. URL: https://doi.org/10.2140/agt.2014.14.597

[18] Paul Thomas Pearson. "The mod 2 Hopf ring for connective Morava K-theory". In: *J. Homotopy Relat. Struct.* 11.3 (2016), pp. 469–491. ISSN: 2193-8407,1512-2891. DOI: 10.1007/s40062-015-0113-z. URL: https://doi.org/10.1007/s40062-015-0113-z

3.4. **Hopf rings in cohomology.** The Hopf rings discussed so far in this document all live quite naturally in homology. However, this is not the only setting in which Hopf rings arise in algebraic topology or homotopy computations. Giusti, Salvatore, and Sinha's computation of the mod-2 cohomology rings of symmetric groups is one such cohomological example. Additionally, Guerra, Salvatore, and Sinha also compute the cohomology rings of extended powers and free infinite loop spaces.

[5] Chad Giusti, Paolo Salvatore, and Dev Sinha. "The mod-2 cohomology rings of symmetric groups". In: *J. Topol.* 5.1 (2012), pp. 169–198. ISSN: 1753-8416,1753-8424. DOI: 10.1112/jtopol/jtr031. URL: https://doi.org/10.1112/jtopol/jtr031

[7] L. Guerra, P. Salvatore, and D. Sinha. *Cohomology rings of extended powers and free infinite loop spaces.* 2023. arXiv: 2304.06435 [math.AT]

3.5. **Hopf algebroids and computations.** As shown by Milnor, the dual Steenrod algebra $H\mathbb{F}_{2*}H\mathbb{F}_2$ is a Hopf algebra [15]. In the C_2-equivariant and motivic settings, the analogous dual Steenrod algebras enjoy the more complicated algebraic structure a Hopf algebroid [10] [25] (see [20, Appendix A1] for more background on Hopf algebroids and homological algebra involving Hopf algebroids).

[20, Appendix A1] Douglas C. Ravenel. *Complex cobordism and stable homotopy groups of spheres.* Vol. 121. Pure and Applied Mathematics. Academic Press, Inc., Orlando, FL, 1986, pp. xx+413. ISBN: 0-12-583430-6; 0-12-583431-4

[10] Po Hu and Igor Kriz. "Real-oriented homotopy theory and an analogue of the Adams-Novikov spectral sequence". In: *Topology* 40.2 (2001), pp. 317–399. ISSN: 0040-9383. DOI: 10.1016/S0040-9383(99)00065-8. URL: https://doi.org/10.1016/S0040-9383(99)00065-8

[25] Vladimir Voevodsky. "Reduced power operations in motivic cohomology". In: *Publ. Math. Inst. Hautes Études Sci.* 98 (2003), pp. 1–57. ISSN: 0073-8301,1618-1913. DOI: 10.1007/s10240-003-0009-z. URL: https://doi.org/10.1007/s10240-003-0009-z

In C_p-equivariant homotopy, where p is an odd prime, the structure of the dual Steenrod algebra is even more richly complicated. For instance, $H\underline{\mathbb{F}}_{p_\star}H\underline{\mathbb{F}}_p$ is not flat over its coefficients

$H\underline{\mathbb{F}}_{p_\star}$ [23]. Additionally, many of the algebraic structure formulas are known through recursive rather than explicit formulas [11], making direct calculations more challenging than in the C_2-equivariant case.

[23] Krishanu Sankar and Dylan Wilson. *On the C_p-equivariant dual Steenrod algebra.* 2021. arXiv: `2103.16006 [math.AT]`

[11] Po Hu et al. *The $\mathbb{Z}/p$-equivariant dual Steenrod algebra for an odd prime p.* 2023. arXiv: `2205.13427 [math.AT]`

Nonetheless, these dual Steenrod algebras serve as useful input into the Adams spectral sequence and thus may be gleaned for information about the stable homotopy of spheres.

3.6. **Hopf algebras and p-polar rings.** Recent work of Tillman Bauer classifies all indecomposable graded, connected, countable-dimensional, commutative and cocomutative Hopf algebras over a perfect field k of characteristic p [3]. Bauer also extends this work to define a graded p-polar ring to be an analog of a graded commutative ring where multiplication is only allowed on p-tuples (instead of pairs) of elements of equal degree [2]. Bauer then shows that the free affine p-adic group scheme functor, as well as the free formal group functor, defined on k-algebras for a perfect field k of characteristic p, factors through p-polar k-algebras. It follows that the same is true for any affine p-adic or formal group functor, in particular for the functor of p-typical Witt vectors. As an application, Bauer shows that the homology of the free E_n-algebra $H_*(\Omega^n\Sigma^n X;\mathbb{F}_p)$, as a Hopf algebra, only depends on the p-polar structure of $H_*(X;\mathbb{F}_p)$ in a functorial way.

[3] Tilman Bauer. *On the structure of abelian Hopf algebras.* 2022. arXiv: `2203.01676 [math.AT]`

[2] Tilman Bauer. *Graded p-polar rings and the homology of $\Omega^n\Sigma^n X$.* 2024. arXiv: `2203.05286 [math.AT]`

3.7. **Other applications.** Here are a few additional papers I have enjoyed or found helpful to read. This is by no means a complete list. If you have a favorite paper you would like to have appear here, please feel free to email me at `sarahllpetersen@gmail.com`.

[14] Nitu Kitchloo and W. Stephen Wilson. "On the Hopf ring for $ER(n)$". In: *Topology Appl.* 154.8 (2007), pp. 1608–1640. ISSN: 0166-8641,1879-3207. DOI: `10.1016/j.topol.2007.01.001`. URL: `https://doi.org/10.1016/j.topol.2007.01.001`

[24] Markus Spitzweck and Paul Arne Ø stvæ r. "Motivic twisted K-theory". In: *Algebr. Geom. Topol.* 12.1 (2012), pp. 565–599. ISSN: 1472-2747,1472-2739. DOI: `10.2140/agt.2012.12.565`. URL: `https://doi.org/10.2140/agt.2012.12.565`

[4] Hood Chatham, Jeremy Hahn, and Allen Yuan. "Wilson spaces, snaith constructions, and elliptic orientations". In: *Invent. Math.* 236.1 (2024), pp. 165–217. ISSN: 0020-9910,1432-1297. DOI: `10.1007/s00222-024-01239-3`. URL: `https://doi.org/10.1007/s00222-024-01239-3`

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[1] Tom Bachmann and Michael J. Hopkins. *Cohomology of motivic Eilenberg–MacLane spaces.* (preprint not yet available). 2022.

[2] Tilman Bauer. *Graded p-polar rings and the homology of* $\Omega^n\Sigma^n X$. 2024. arXiv: 2203.05286 [math.AT].

[3] Tilman Bauer. *On the structure of abelian Hopf algebras.* 2022. arXiv: 2203.01676 [math.AT].

[4] Hood Chatham, Jeremy Hahn, and Allen Yuan. "Wilson spaces, snaith constructions, and elliptic orientations". In: *Invent. Math.* 236.1 (2024), pp. 165–217. ISSN: 0020-9910,1432-1297. DOI: 10.1007/s00222-024-01239-3. URL: https://doi.org/10.1007/s00222-024-01239-3.

[5] Chad Giusti, Paolo Salvatore, and Dev Sinha. "The mod-2 cohomology rings of symmetric groups". In: *J. Topol.* 5.1 (2012), pp. 169–198. ISSN: 1753-8416,1753-8424. DOI: 10.1112/jtopol/jtr031. URL: https://doi.org/10.1112/jtopol/jtr031.

[6] Paul G. Goerss. "Hopf rings, Dieudonné modules, and $E_*\Omega^2 S^3$". In: *Homotopy invariant algebraic structures (Baltimore, MD, 1998).* Vol. 239. Contemp. Math. Amer. Math. Soc., Providence, RI, 1999, pp. 115–174. ISBN: 0-8218-1057-X. DOI: 10.1090/conm/239/03600. URL: https://doi.org/10.1090/conm/239/03600.

[7] L. Guerra, P. Salvatore, and D. Sinha. *Cohomology rings of extended powers and free infinite loop spaces.* 2023. arXiv: 2304.06435 [math.AT].

[8] Michael A. Hill and Michael J. Hopkins. *Real Wilson Spaces I.* 2018. arXiv: 1806.11033 [math.AT].

[9] M. J. Hopkins and J. R. Hunton. "On the structure of spaces representing a Landweber exact cohomology theory". In: *Topology* 34.1 (1995), pp. 29–36. ISSN: 0040-9383. DOI: 10.1016/0040-9383(94)E0013-A. URL: https://doi.org/10.1016/0040-9383(94)E0013-A.

[10] Po Hu and Igor Kriz. "Real-oriented homotopy theory and an analogue of the Adams-Novikov spectral sequence". In: *Topology* 40.2 (2001), pp. 317–399. ISSN: 0040-9383. DOI: 10.1016/S0040-9383(99)00065-8. URL: https://doi.org/10.1016/S0040-9383(99)00065-8.

[11] Po Hu et al. *The* $\mathbb{Z}/p$*-equivariant dual Steenrod algebra for an odd prime* p. 2023. arXiv: 2205.13427 [math.AT].

[12] John R. Hunton and Paul R. Turner. "Coalgebraic algebra". In: *J. Pure Appl. Algebra* 129.3 (1998), pp. 297–313. ISSN: 0022-4049,1873-1376. DOI: 10.1016/S0022-4049(97)00076-5. URL: https://doi.org/10.1016/S0022-4049(97)00076-5.

[13] John R. Hunton and Paul R. Turner. "The homology of spaces representing exact pairs of homotopy functors". In: *Topology* 38.3 (1999), pp. 621–634. ISSN: 0040-9383. DOI: 10.1016/S0040-9383(98)00037-8. URL: https://doi.org/10.1016/S0040-9383(98)00037-8.

[14] Nitu Kitchloo and W. Stephen Wilson. "On the Hopf ring for $ER(n)$". In: *Topology Appl.* 154.8 (2007), pp. 1608–1640. ISSN: 0166-8641,1879-3207. DOI: 10.1016/j.topol.2007.01.001. URL: https://doi.org/10.1016/j.topol.2007.01.001.

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[16] Paul Thomas Pearson. "The connective real K-theory of Brown-Gitler spectra". In: *Algebr. Geom. Topol.* 14.1 (2014), pp. 597–625. ISSN: 1472-2747,1472-2739. DOI: 10.2140/agt.2014.14.597. URL: https://doi.org/10.2140/agt.2014.14.597.

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[19] Sarah Petersen. *The $H\underline{\mathbb{F}}_2$-homology of C_2-equivariant Eilenberg-MacLane spaces.* 2022. arXiv: 2206.08165 [math.AT].

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[23] Krishanu Sankar and Dylan Wilson. *On the C_p-equivariant dual Steenrod algebra.* 2021. arXiv: 2103.16006 [math.AT].

[24] Markus Spitzweck and Paul Arne Ø stvæ r. "Motivic twisted K-theory". In: *Algebr. Geom. Topol.* 12.1 (2012), pp. 565–599. ISSN: 1472-2747,1472-2739. DOI: 10.2140/agt.2012.12.565. URL: https://doi.org/10.2140/agt.2012.12.565.

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