(*Preliminary draft*)

Open and In Progress Research Questions
eCHT Research Workshop on Hopf Rings
June 24 - 28, 2024

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Introduction

This compilation of open and in progress research questions on the topics of Hopf algebras, rings, and algebroids appearing in algebraic topology is meant for discussion during the eCHT Research Workshop on Hopf Rings running from 11 AM - 2 PM Eastern Time on June 24 - 28, 2024. A wide range of problems in terms of difficulty, specificity, and topic are described within this document. It is encouraged to skim the entire document and focus on the questions that most interest you. Problems are generally grouped by topic:

Research Questions:

1. Ravenel–Wilson Hopf Ring Techniques

(*Contributed by Jack Carlisle, Sarah Petersen, and Steve Wilson*)

I. **Overarching goal**: For each ring spectrum, compute the Hopf ring of the homology of the (non negatively-graded) spaces representing its omega spectrum (see [30, 24, 25, 17, 18, 31, 32, 22] for examples).
   A. Go further: Understand the extent to which Hopf ring structures exist in the homology of the (non negatively-graded)spaces representing the omega spectra of ring spectra in additional categories such as C_2-equivariant and motivic spectra.
      1. The story in C_2-equivariant and motivic spectra is more complicated as one generally does not have a Künneth isomorphism.
      2. Some important examples, such as the $H\underline{\mathbb{F}}_2$-homology of C_2-equivariant Eilenberg–MacLane spaces [22], and the motivic cohomology of motivic Eilenberg–MacLane spaces as well as the Hopf ring of algebraic cobordism [1] enjoy the structure of a Hopf ring.
      3. Some outstanding questions related to the $H\underline{\mathbb{F}}_2$-homology of C_2-equivariant Eilenberg–MacLane spaces include:
         a. The mod p-homology of classical nonequivariant Eilenberg–MacLane spaces has a global description. Specifically, Ravenel and Wilson show

[1]To appear

Theorem 1 (Ravenel-Wilson [30]). *H_*K_* is the free Hopf ring on $H_*K_0 = H_*[\underline{\mathbb{F}}_p]$, H_*K_1, and $H_*\mathbb{C}P^\infty \subset H_*K_2$ subject to the relation that $e_1 \circ e_1 = \beta_1$.*

Can a similar statement be obtained in the C_2-equivariant case? If so, what specifically, is the global structure of the Hopf rings that do arise? One may also ask how the Hopf rings here relate to Hill and Hopkins' work extending Ravenel and Wilson's construction of a universal Hopf ring over MU^* to C_2-equivariant homotopy theory [12].

b. How does the stable relation

$$\tau_i^2 = (u + a\tau_0)\xi_{i+1} + a\tau_{i+1}$$

in the C_2-equivariant dual Steenrod algebra

$$\mathcal{A}_\star^{C_2} = H\underline{\mathbb{F}}_2[\tau_0, \tau_1, \cdots, \xi_1, \xi_2, \cdots]/(\tau_i^2 = (u + a\tau_0)\xi_{i+1} + a\tau_{i+1})$$

arise from the unstable calculation of $H_\star K_V$?

c. Is it possible to explicitely compute the differentials in the twisted bar spectral sequences computing $H_\star K_{*\sigma}$, perhaps in terms of some norm-like structure?

B. Where all can the Hopf ring structure in the bar spectral sequence be applied?
  1. The Hopf ring structure in the bar spectral sequence significantly restricts potential differentials. For example, differentials must hit primitives in filtrations 0, 1, or 2 (see [30, Sections 7 and 8], [29], and [32, Section 3] for an introduction).
  2. This Hopf ring structure in the bar spectral sequence has been used to compute the mod p-homology of nonequivariant Eilenberg–MacLane spaces [30, Section 8], the Morava K-theory of Eilenberg–MacLane spaces [25], and the $H\underline{\mathbb{F}}_2$-homology of C_2-equivariant Eilenberg–MacLane spaces [22].

C. Can we compute $K(n)_*QS^0$, the Morava K-theory of the sphere spectrum (regarded as a ring spectrum)? Better yet, can we compute the Hopf ring of the Morava K-theory of all the spaces representing the omega spectrum QS^0?

D. Can we compute $K(n)_*ER(2)_*$, the Morava K-theory of the spaces representing the Johnson–Wilson theory $ER(2)$? (We know the mod 2-homology of these spaces [18].) Note the Morava K-theory of $ER(1)$ is known since its simply BO [17].

E. Find the names for algebra generators given in [17, 31, 32], for instance. Motivation: names better than "x" are useful in applications.

F. What can we say about the $H\underline{\mathbb{F}}_2$-homology of the zeroth space of $MU_\mathbb{R}$, Real cobordism? Can we compute all or some of $H\underline{\mathbb{F}}_{2\star}^{C_2} MU_{\mathbb{R}\star}$? Do these computations carry any Hopf structures?

II. **Geometric cobordism and equivariant cobordism of maps**. This is an ongoing research project led by Jack Carlisle and Sarah Petersen. We are looking for up to three (junior) researchers to join our project. Everyone is welcome to think about it during the workshop. If you are interested in collaborating after the workshop, please email Jack Carlisle and Sarah Petersen at `jcarlisl@nd.edu` and `sarahllpetersen@gmail.com` to express interest.

A. **Project description**: Let A be an abelian group. The cobordism ring of A-manifolds (unoriented, without boundary) is represented by the A-spectrum mO_A, whose Vth space (for V an orthogonal A-representation) is

$$mO_A(V) = \mathrm{Thom}\left(\gamma \to \mathrm{Gr}_{\dim V}(V \oplus \mathbb{R}^\infty)\right).$$

This project aims to prove the cobordism group of A-equivariant maps between A-manifolds is the mO_A-homology of a space representing mO_A, similar to the case for complex cobordism, where the cobordism group of complex maps of codimension $2n$ is the complex bordism group MU_*MU_n, and MU_*MU_* is the cobordism group of all maps with even codimension (see [24, discussion between Theorem 2.1 and Proposition 2.2]).

B. This project is closely related to work of Stong [28, 27], Rinne [26], and Hara [11, 10].

2. Hopf Rings and Dieudonné Theory

(*Contributed by Tilman Bauer and Sarah Petersen*)

III. To appear (*Contributions from Tilman Bauer*)

IV. **Related equivariant algebra questions** (*Contributed by Sarah Petersen*)

A. Classically, the Witt vectors (or rather a dual notion) play a fundamental role in the Dieudonné theory associating a category of modules over some ring to category of graded, connected, bicommutative Hopf algebras over $\mathbb{F}_p$. [8]. In equivariant algebra, there are several definitions of Witt vectors, each corresponding to a definition of equivariant topological Hochschild homology (see [2], for instance).

1. What are the various types/definitions of equivariant Witt vectors?
2. Where do they show up in equivariant algebra?
3. Are any of these related to Hopf ring (like) structures appearing in equivariant Bredon homology?

V. **Computations relating to Landweber exactness** (*Contributed by Sarah Petersen*)

A. Nonequivariantly, Dieudonné theory, together with Ravenel's computation of $BP_*\Omega^2S^3$ [23], can be used to deduce the results of Hopkins–Hunton [13] and Hunton–Turner [15, 16] on the homology of the spaces representing a Landweber exact theory [8, Sections 10 and 11]. Equivariantly, in the $RO(C_2)$, or more generally $RO(G)$-graded setting, there is no obvious Dieudonné theory generalization, owing to the fact that maps between free $E_\star$-modules, where E is a G-ring spectrum may not have free $E_\star$-kernels or cokernels. Is there a way to algebraicize and/or categorify this?

B. Carrick–Guillou–Petersen compute $BP\mathbb{R}_\star\Omega^\rho S^{\rho+1}$ (*in progress*), the C_2-equivariant analogue of $BP_*\Omega^2S^3$. Thus we can ask what the equivariant computation of $BP\mathbb{R}_\star\Omega^\rho S^{\rho+1}$ tells us, if anything, about the C_2-equivariant homology of C_2-equivariant analogues of Landweber exact spectra when the underlying nonequivariant computations are understood in the language of classical Dieudonné theory.

VI. **Equivariant Brown–Gitler spectra** (*Contributed by Sarah Petersen*)

Classically, one of the main reasons it is practical to compute Hopf rings in homology by studying the associated ring object in Dieudonné modules is that there is close relationship between Brown–Gitler spectra and Dieudonné modules (see [8, 20, 19, 21]). It would be interesting to know to what extent a similar program can be carried out equivariantly.

A. One way of constructing Brown–Gitler spectra is via Brown representability and observing that the functor $E \to DH_*\Omega^\infty E$ taking a ring spectrum to the Dieudonné ring associated to the Hopf algebra $H_*\Omega^\infty E$ is exact (see [7, 6]). Similarly, in the C_2-equivariant world, is the functor $X \to DH_{*+b\sigma}\Omega^\infty X$ exact for each b? If it is exact, what are the properties of the represnting spectrum?

B. Classically, Brown–Gitler spectra can also be constructed from the Snaith summands of $\Omega^2 S^3$. This was proven by Brown–Peterson at the prime $p = 2$ [3], partially for odd primes by Ralph Cohen [5], and finally in all cases by Hunter–Kuhn [14]. In the C_2-equivariant setting, do the Snaith summands of $\Omega^\rho S^{\rho+1}$ deserve to be Brown–Gitler spectra? What are their comodules? Can we characterize them homotopically?

C. Non-equivariantly, two conditions characterize Brown-Gitler spectra. First, these spectra realize certain sub-comodules of the dual Steenrod algebra. Additionally, they satisfy a surjectivity condition coming from the geometry involved in Brown–Gitler's original construction and witnessed by a certain stage in their Postnikov construction. It is not currently known if a C_2-equivariant analogue of the surjectivity condition [4, Theorem 1.1 (ii)] holds or if there should be some other criterion determining (integral) Brown–Gitler spectra in the C_2-equivariant case. What homotopically characterizes C_2-equivariant Brown–Gitler spectra?

D. Can one construct C_2-equivariant Brown–Gitler spectra analogously to Brown and Gitler's original construction in [4]? Since the C_2-equivariant dual Steenrod algebra is a Hopf algebroid rather than a Hopf algebra, there is no obvious definition of the antipode s on the C_2-equivariant Steenrod algebra. What happens if one uses Brown–Comenetz duality to define a map playing the role of s? If such spectra are constructed, do they have useful computational properties? J.D. Quigley and others have been thinking about this problem, particularly in the motivic setting.

E. Note: David Chan and Sarah Petersen have work in progress using a Thom spectrum construction for C_p-equivariant Brown–Gitler spectra at all primes p.

3. Hopf algebras in the group cohomology of arithmetic groups

(*Contributed by Peter Patz*)

VII. To appear

4. Hopf objects and broader structures

(*Contributed by Peter May*)

VIII. Hopf rings and E_∞-structures

A. Given a space X, Ravenel–Wilson Hopf rings techniques restrict attention to the component of $\Omega^\infty\Sigma^\infty X$. In this sense, these Hopf rings only see a small portion of the total structure on $H_*\Omega^\infty\Sigma^\infty X$ when we really also have all Dyer–Lashof operations. Here are some related questions.

1. What additional information can we retain by studying all components of $\Omega^\infty\Sigma^\infty X$ from a Hopf ring perspective when X is not connected?
2. What can one see by looking at Dyer–Lashof operations on all components?
3. Tyler Lawson used secondary power operations to show that the 2-primary Brown–Peterson spectrum does not admit an E_n-structure for any $n \geq 12$. Are there primary operations in homology, perhaps visible in the full structure of Hopf rings, that one can use to deduce the same result?

IX. Hopf algebras as groups in the category of cocommutative coalgebras

A. In the category of cocommutative coalgebras, the tensor product is the cartesian monoidal product. Thus Hopf algebras that are cocommutative coalgebras are groups in this category. Thus we have notions such as fixed point groups and can seriously look at doing group theory in this setting. A paper by Guillou, May, and Rubin is a starting point for this perspective [9]. Initial questions include:

1. Analyze the subgroups corresponding to finite subalgebras of the Steenrod algebra.
2. Is there an algebraic chromatic homotopy theory defined by these subgroups, that is subgroups of the Steenrod algebra as a Hopf algebra?

B. What about ring theory in the category of cocomutative coalgebras?

1. Cocommutativity plays a critical role in this analysis of Hopf algebras as groups. What role does cocomoutativity play in the ring structure?
2. More generally, what ring theoretic statements can be made in this category? Are there useful applications?

5. Hopf Rings and Geometric Representatives

(*Contributed by Dev Sinha*)

X. **The Geometry of Eilenberg–MacLane spaces**. This is an open and ongoing a research project lead by Dev Sinha. Hanna Hoffman, Dana Hunter, Kristine Pelatt, Sarah Petersen and Courtney Thatcher have all contributed at various points. If you are interested in collaborating after the workshop, please email Dev Sinha at `dps@uoregon.edu`.

A. **Project description**. The goals of this project are to write a largely expository paper describing Cartan's computation of the homology of Eilenberg–MacLane spaces and its relationship to the homological Leray-Serre spectral sequence, writing up a Divided powers Hopf ring presentation of the same homology, giving a basis in terms of graphs, and comparing all of these with the Ravenel–Wilson computation. Additionally, all computations should be given fully geometric descriptions (in terms of generating elements) and be readily accessible, especially to students.

XI. Compute the Morava K-theory of symmetric groups.

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