This document is posted with the permission of the author Peter May. It was hand-written by Peter, most likely in 1963 or 1964. It describes the cohomology of $A(2)$, which later came to be known as the Adams E_2-page for *tmf*. The computation is correct in all respects (as far as I have noticed). It gives a complete roadmap for the May spectral sequence, including all differentials and all hidden multiplicative structure. Although it was never published, this document slightly precedes the work of Shimada and Iwai that was published in 1967 (and received by the journal in 1966).

This document represents a very early computational hint of the existence of a spectrum that would be constructed much later, and even later be named topological modular forms.

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$p = 2$ $A_n \subset A$ generated by $\{Sq^1, \ldots, Sq^{2^n}\}$ ($= A(n+1)$ — my paper, chapter I)

$E^0 A_n = V L_n$ where L_n has basis $\{P^i_j \mid i+j \le n+1\}$, $\bar{X}(L_n)^* = P\{R^i_j \mid i+j \le n+1\}$ with

$d R^i_j = c_{ij} = \sum R^i_k R^{i+k}_{j-k}$. Higher differentials computable by pull-back from $E_r A$.

(1) $A_0 = E\{Sq^1\}$ hence $H^*(A_0) = P\{h_0\}$

(2) $H^*(E^0 A_1) = P\{h_0, h_1, b^0_2\}/(h_0 h_1)$ and $d_2(b^0_2) = h_1^3$ since $b^0_2 = \langle h_1, h_0, h_1, h_0 \rangle$. Then

$E_\infty = E_3 = P\{h_0, h_1, \chi, \omega\}/(h_0 h_1, h_1^3, h_1\chi, \chi^2 + h_0^2\omega) = H^*(A_1)$, where we have

$h_0 \in (1,1)$, $h_1 \in (1,2)$, $\chi = h_0 b^0_2 = \langle h_0, h_1, h_1^2 \rangle \in (3,7)$, $\omega = (b^0_2)^2 = \langle h_1^2, h_1, h_1^2, h_1 \rangle \in (4,12)$.

Thus $H^*(A_1) = P\{\omega\} \otimes B\{h_0^i, h_0^i\chi, h_1, h_1^2 \mid i \ge 0\}$ as a left $P\{\omega\}$-module.

(3) $H^*(E^0 A_2) = P\{h_0, h_1, h_2, h_0(1), b^0_2, b^1_2, b^0_3\}/I$ where I is the ideal generated by

$h_0 h_1, h_1 h_2, h_0 h_0(1) + h_2 \underline{b^0_2}, h_2 h_0(1) + h_0 b^1_2, h_0(1)^2 + h_1^2 b^0_3 + b^1_2 b^0_2$ (hence $h_0^2 b^1_2 + h_2^2 b^0_2 \in I$).

$h_0(1) \leftrightarrow R^0_2 R^1_2 + R^1_1 R^0_3$

Thus $E_2 A_2$ admits the following Z_2-basis ($a > 0$, $l \ge 0$, $b \ge 0$, $c \ge 0$, $k \ge 0$, $i \ge 0$, $j \ge 0$, $\varepsilon = 0$ or 1)

(i) $(b^1_2)^b (b^0_3)^c h_0^i h_2^j$, $i+j > 0$

(ii) $(b^1_2)^b (b^0_3)^c h_0^{2l}(1) h_0^i h_2^\varepsilon$, $i+\varepsilon > 0$, $l > 0$

(iii) $(b^0_2)^a (b^0_3)^c h_0^{2l}(1) h_0^i h_2^\varepsilon$, $i+\varepsilon > 0$, $a > 0$

(iv) $(b^1_2)^b (b^0_3)^c h_1^k h_0^l(1)$

(v) $(b^0_2)^a (b^0_3)^c h_1^k h_0^l(1)$, $a > 0$

$d_2(b^0_2) = h_1^3 + h_0^2 h_2$, $d_2(b^1_2) = h_2^3$, $d_2(b^0_3) = h_1 b^1_2$, and $d_2 h_0(1) = h_0 h_2^2$. Let

$\omega = (b^0_2)^2$, $\bar{\beta} = (b^0_3)^2$, $a = h_1 h_0(1)$, $b = h_0^2(1)$, $c = b^1_2 h_0(1)$, $d = (b^1_2)^2$, $\alpha = h_0 b^0_3$, $\nu = h_2 b^0_3$;

writing out d_2 in terms of our basis, we find that E_3 admits as basis:

(a) $P\{\omega, d, \bar{\beta}\} \otimes E\{b\} \otimes B\{1, h_0, h_1, h_2, h_0 h_2, h_2^2, c, h_1 c, \alpha, \nu, h_0\nu\}$

(b) $P\{\omega, d, \bar{\beta}\} \otimes B\{h_0\alpha, h_2\nu\}$

(c) $P\{\omega, \bar{\beta}\} \otimes B\{h_1^2, h_1^3, a, h_1 a, h_0^i, h_0^i\alpha \mid i \ge 2\}$

$d_4(\bar{\beta}) = h_2 d$ and using the relations in E_3, we find that $E_5 = E_\infty$ admits as basis:

(d) $P\{\omega, d, \beta\} \otimes E\{b\} \otimes B\{1, c, \alpha, \nu, \alpha\nu, \nu^2, \gamma, \gamma c\}$

(e) $P\{\omega, \beta\} \otimes B\{h_0^{i+1}, h_0^{i+1}\alpha, h_0^{i}\alpha^2, h_0^{i}\alpha^3, h_1, h_1^2, h_2, h_0h_2, h_0^2h_2, h_2^2, a, h_1a, h_0b, h_0\nu, h_0^2\nu, h_0c, h_0^2c,$

$h_2\nu, h_0h_2\nu, h_0d, h_0^2d, h_1d, h_0\nu b, h_0\alpha d, h_0^2\alpha d, h_0bd, h_1\gamma, h_1^2\gamma, \delta, h_1\delta \mid i \geq 0\}$

where $\beta = \bar{\beta}^2 = (b_3^0)^4$, $\gamma = h_1\bar{\beta} = h_1(b_3^0)^2$, and $\delta = a\bar{\beta} = h_1h_0(1)(b_3^0)^2$. Thus E_∞ is generated as an algebra by $\{\omega, d, \beta, h_0, h_1, h_2, a, b, c, \alpha, \nu, \gamma, \delta\}$ where

$h_i \in (1, 2^i)$, $i = 0, 1,$ and 2, and

$\omega = \left\langle (h_1^2, h_0), \begin{pmatrix} h_1 & h_0 \\ h_0h_2 & h_1^2 \end{pmatrix}, \begin{pmatrix} h_1^2 & h_0 \\ h_0h_2 & h_1 \end{pmatrix}, \begin{pmatrix} h_1 \\ h_0h_2 \end{pmatrix} \right\rangle \in (4, 12)$

$d = \langle h_2, h_2^2, h_2, h_2^2 \rangle = \langle h_2, h_2^2, h_2^2, h_2 \rangle \quad \in (4, 24)$

$\beta = \langle h_2, d, h_2, d \rangle \quad \in (8, 56)$

$a = \langle h_1, h_0, h_2^2 \rangle \quad \in (3, 11)$

$b = \langle h_0, h_2^2, h_0, h_2^2 \rangle = \langle h_0, h_2^2, h_2^2, h_0 \rangle \quad \in (4, 18)$

$c = \langle h_0, h_2^2, h_2, h_2^2 \rangle = \langle h_0, h_2^2, h_2^2, h_2 \rangle \quad \in (4, 21)$

$\alpha = \langle h_0, h_1, h_2, h_2^2 \rangle \quad \in (3, 15)$

$\nu = \langle h_2, h_1, h_2, h_2^2 \rangle \quad \in (3, 18)$

$\gamma = \langle h_1, h_2, d \rangle \quad \in (5, 30)$

$\delta = \langle a, d, h_2 \rangle \in \langle a, h_2, d \rangle$ (αd interchanging) $\in (7, 39)$

A defining system of relations for E_∞ is the following:

$h_0h_1 = 0, h_1h_2 = 0, h_0^2h_2 = h_1^3, h_0h_2^2 = 0, h_2^3 = 0;$

$h_0a = 0, h_1^2a = 0, h_2a = 0, a^2 = 0, ab = 0, ac = 0, ad = 0, \alpha a = 0, \nu a = 0;$

$h_0^2b = h_2^2\omega, h_1b = h_0^2\nu, h_2b = h_0c, h_0\alpha b = h_2\nu\omega, \alpha^2b = \nu^2\omega, \nu b = \alpha c, b^2 = \omega d, bd = c^2;$

$h_1c = h_2^2\alpha, h_0d = h_2c, h_1d = h_2^2\nu, h_2d = 0, \alpha d = \nu e;$

$h_1\alpha = 0, h_1\nu = 0, h_2\alpha = h_0\nu, h_2\alpha^2 = h_1^2\gamma, h_0\nu^2 = 0, h_2\nu^2 = 0;$

$h_0\gamma = 0, h_2\gamma = 0, \gamma a = h_1\delta, \gamma b = \alpha^2\nu, \gamma c = \alpha\nu^2, \gamma d = \nu^3, \alpha\gamma = 0, \nu\gamma = 0, \gamma^2 = h_1^2\beta;$

$h_0\delta = 0, h_1^2\delta = 0, h_2\delta = 0, \delta a = 0, \delta b = 0, \delta c = 0, \delta d = 0, \alpha\delta = 0, \nu\delta = 0, \gamma\delta = h_1a\beta, \delta^2 = 0, \alpha^4 = h_0^4\beta$

$H^*(A_2)$ has generators $\{\omega, \alpha_1', \nu, \beta, \gamma, \delta, a, b, c, d, h_0, h_1, h_2\}$ given by the above Massey products. Clearly $\langle h_0, h_1, h_0\rangle = h_1^2$, $\langle h_1, h_0, h_1\rangle = h_0 h_2$, and $\langle h_1, h_2, h_1\rangle = h_2^2$. Thus

$h_1 \langle h_0, h_1, \alpha\rangle = \langle h_1, h_0, h_1\rangle \alpha = h_0 h_2 \alpha = h_1 b \Rightarrow b = \langle h_0, h_1, \alpha\rangle$

$h_1 \langle h_2, h_1, \alpha\rangle = \langle h_1, h_2, h_1\rangle \alpha = h_2^2 \alpha = h_1 c \Rightarrow c = \langle h_2, h_1, \alpha\rangle$

$h_1 \langle h_0, h_1, \nu\rangle = \langle h_1, h_0, h_1\rangle \nu = h_0 h_2 \nu = h_1 c \Rightarrow c = \langle h_0, h_1, \nu\rangle$

$h_1 \langle h_2, h_1, \nu\rangle = \langle h_1, h_2, h_1\rangle \nu = h_2^2 \nu = h_1 d \Rightarrow d = \langle h_2, h_1, \nu\rangle$

We use this to check all dimensionally possible non-trivial extensions.

$\alpha c = \alpha \langle h_1, h_0, h_2^2\rangle = \langle \alpha, h_1, h_0\rangle h_2^2 = b h_2^2 = h_0^2 d$

$\alpha \gamma = \alpha \langle h_1, h_2, d\rangle = \langle \alpha, h_1, h_2\rangle d = cd$

$\alpha_1^4 \gamma = \alpha_1^3 cd = \gamma d^2 \omega \neq 0$ but $h_0^4 \gamma\beta = 0 \Rightarrow \alpha_1^4 = h_0^4 \beta + g^2 \omega$

$\nu\gamma = \nu \langle h_1, h_2, d\rangle = \langle \nu, h_1, h_2\rangle d = d^2$

$\nu\gamma^2 = \gamma d^2 \neq 0$ but $\nu h_1^2 \beta = 0 \Rightarrow \gamma^2 = h_1^2 \beta + \nu^2 d$

Since $d_2(b_2^\circ h_0(1)) = h_1^3 h_0(1)$ and $\{h_2 b_2^\circ h_0(1)\} = \{h_0 b\}$, we have $\langle h_1 a, h_1, h_2\rangle = h_0 b$, hence

$h_1^2 \delta = h_1 a \gamma = h_1 a \langle h_1, h_2, d\rangle = \langle h_1 a, h_1, h_2\rangle d = h_0 b d$. Recall that $\delta = \langle h_2, d, a\rangle$;

since $h_0 a = 0$ in E_2 and $d_2(b_3^\circ c) = ad$, $\langle d, a, h_0\rangle = \alpha c = \nu b$, and therefore

$h_0 \delta = \langle h_2, d, a\rangle h_0 = h_2 \langle d, a, h_0\rangle = h_2 \nu b = h_0 \alpha d$.

$\gamma\delta d = 0 = h_1 a \beta d$ but $cd^3 \neq 0$, hence $\gamma\delta = h_1 a \beta$;

$\alpha\delta = \langle h_0, h_1, h_2, h_2^2\rangle \delta \subset \langle h_0, h_1, h_2, 0\rangle = 0$ (strictly defined, zero indeterminacy)

$\nu\delta = \langle h_2, d, a\rangle \nu = h_2 \langle d, a, \nu\rangle = 0$

$b\delta = h_2 \langle d, a, b\rangle = 0$

$c\delta = h_2 \langle d, a, c\rangle = 0$

$d\delta = h_2 \langle d, a, d\rangle = 0$

$\delta^2 = h_2 \langle d, a, \delta\rangle = 0$

Since $w(h_0 a) = 6 < 8 = w(\omega)$, $h_0 a = 0$; since $w(\alpha d) = w(\nu c) = 15 < 17 = w(\delta)$, $\alpha d = \nu c$. there are no further possible non-trivial extensions.

Thus $H^*(A_2)$ admits the following defining set of relations:

(i) $h_0 h_1 = 0,\ h_1 h_2 = 0,\ h_0^2 h_2 = h_1^3,\ h_0 h_2^2 = 0,\ h_2^3 = 0$

(ii) $h_0 a = 0,\ h_1^2 a = 0,\ h_2 a = 0,\ a^2 = 0,\ ab = 0,\ ac = 0,\ ad = 0,\ a\alpha = h_0^2 d;\ a\nu = 0$

(iii) $h_0^2 b = h_2^2 w,\ h_1 b = h_0^2 \nu,\ h_2 b = h_0 c,\ b^2 = wd,\ bd = c^2,\ h_0 \alpha b = h_2 \nu w,\ \alpha^2 b = \nu^2 w,\ \nu b = \alpha c;$

(iv) $h_1 c = h_2^2 \alpha,\ h_2 c = h_0 d,\ \nu c = \alpha d,\ h_1 d = h_2^2 \nu,\ h_2 d = 0,\ cd = \alpha\gamma,\ d^2 = \nu\gamma;$

(v) $h_1 \alpha = 0,\ h_1 \nu = 0,\ h_2 \alpha = h_0 \nu,\ h_0 \nu^2 = 0,\ h_2 \nu^2 = 0,\ \alpha^4 = h_0^4 \beta + d^2 w$

(vi) $h_0 \gamma = 0,\ h_1^2 \gamma = h_2 \alpha^2,\ h_2 \gamma = 0,\ a\gamma = h_1 \delta,\ b\gamma = \alpha^2 \nu,\ c\gamma = \alpha \nu^2,\ d\gamma = \nu^3,\ \gamma^2 = h_1^2 \beta + \nu^2 g;$

(vii) $h_0 \delta = h_0 \alpha d,\ h_1^2 \delta = h_0 b d,\ h_2 \delta = 0,\ a\delta = 0,\ b\delta = 0,\ c\delta = 0,\ d\delta = 0,\ \alpha\delta = 0,\ \nu\delta = 0,\ \gamma\delta = h_1 a \beta,\ \delta^2 = 0$