MAT 7570
The Mayer-Vietoris Sequence

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Abstract

This is an expository paper that aims to provide an overview of the Mayer-Vietoris sequence. The intended reader is a student enrolled in a first course in algebraic topology. We first state and prove the main theorem, consider other versions of the sequence, and discuss its relation to other ideas such as homology theories. Then, we work through different examples of computing homology groups by applying the sequence and discuss its generalization to the Mayer-Vietoris spectral sequences.

1 Main Results

In this section, we state and prove the main result that yields the Mayer-Vietoris sequence for ordinary unreduced homology. Then, we state both a version of the Mayer-Vietoris sequence for reduced homology and a version for cohomology. Finally, we discuss how the Mayer-Vietoris sequence is related to the van Kampen Theorem and homology theories.

1.1 Statement & Proof

Theorem 1.1.1 (Mayer-Vietoris Sequence)**.** Suppose X is a topological space such that $X = \mathrm{int}(A) \cup \mathrm{int}(B)$, where A and B are subspaces of X and $\mathrm{int}(A)$ and $\mathrm{int}(B)$ denote the interiors of A and B, respectively. Then we obtain the following long exact sequence :

$$\cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to H_{n-1}(A \cap B) \to \cdots \to H_0(X) \to 0.$$

This is known as the *Mayer-Vietoris sequence.*

Proof. Let $C_n(X)$ denote the free abelian group generated by the set of singular n-simplices in X, and consider its subgroup $C_n(A + B)$ consisting of elements that are sums of chains in A and chains in B. We have the chain complex $(C_\bullet(X), \delta_\bullet)$ given by

$$\cdots \xleftarrow{\delta_0} C_0(X) \xleftarrow{\delta_1} C_1(X) \xleftarrow{\delta_2} C_2(X) \xleftarrow{\delta_3} \cdots ,$$

where $\delta_n : C_n(X) \to C_{n-1}(X)$ is the boundary map. This map restricts to $C_n(A+B)$ and maps $C_n(A+B)$ to $C_{n-1}(A+B)$. With this, we also have the chain complex $(C_\bullet(A+B), \delta_\bullet)$ given by

$$\cdots \xleftarrow{\delta_0} C_0(A+B) \xleftarrow{\delta_1} C_1(A+B) \xleftarrow{\delta_2} C_2(A+B) \xleftarrow{\delta_3} \cdots .$$

Now, the sequence of inclusion maps $\iota_n : C_n(A+B) \hookrightarrow C_n(X)$ for each n defines a chain map (and, more particularly, a chain homotopy equivalence[1]) between the chain complexes $(C_\bullet(X), \delta_\bullet)$ and $(C_\bullet(A+B), \delta_\bullet)$. In turn, we obtain isomorphisms in homology groups $H_n(A+B) \cong H_n(X)$ for all n.

Next, for $n \geq 0$, consider the sequence

$$0 \to C_n(A \cap B) \xrightarrow{\alpha} C_n(A) \oplus C_n(B) \xrightarrow{\beta} C_n(A+B) \to 0,$$

[1] That this is a chain homotopy equivalence is the subject of Proposition 2.21 in [5].

where $\alpha : x \longmapsto (x, -x)$ and $\beta : (x, y) \longmapsto x + y$. α is injective as it has trivial kernel, and β is surjective by our definition of $C_n(A + B)$. $(\beta \circ \alpha)(x) = 0 \Rightarrow \operatorname{Im}(\alpha) \subset \ker(\beta)$. Lastly, if $(x, y) \in \ker(\beta)$, then $x \in C_n(A)$, $y \in C_n(B)$ and $x = -y \Rightarrow x \in C_n(A \cap B)$ and $\alpha(x) = (x, -x) = (x, y) \Rightarrow ker(\beta) \subset \operatorname{Im}(\alpha)$. Therefore, this is a short exact sequence. Finally, applying the long exact sequence in homology[2] to the corresponding short exact sequence of chain complexes, using the aforementioned isomorphism in homology groups $H_n(A + B) \cong H_n(X)$ for all n, and using that homology commutes with direct sums, the Mayer-Vietoris sequence follows.

□

Remark 1.1.2. We shall briefly note the maps between the homology groups included in the Mayer-Vietoris sequence. The map $\phi : H_n(A \cap B) \to H_n(A) \oplus H_n(B) : x \longmapsto (f(x), g(x))$ involves the maps on homology $f : H_n(A \cap B) \to H_n(A)$ and $g : H_n(A \cap B) \to H_n(B)$ induced from the inclusions $A \cap B \hookrightarrow A$ and $A \cap B \hookrightarrow B$. The map $\psi : H_n(A) \oplus H_n(B) \to H_n(X) : (y, z) \longmapsto i(y) - j(z)$ involves the maps on homology $i : H_n(A) \to H_n(X)$ and $j : H_n(B) \to H_n(X)$ induced from the inclusions $A \hookrightarrow X$ and $B \hookrightarrow X$. Finally, the map $\delta : H_n(X) \to H_{n-1}(A \cap B)$ is the usual boundary map.

1.2 Other Versions

Theorem 1.2.1 (Mayer-Vietoris Sequence for Reduced Homology)**.** Suppose X is a topological space such that $X = \operatorname{int}(A) \cup \operatorname{int}(B)$, where A and B are subspaces of X such that $A \cap B \neq \emptyset$ and $\operatorname{int}(A)$ and $\operatorname{int}(B)$ denote the interiors of A and B, respectively. Then we obtain the following long exact sequence in reduced homology :

$$\cdots \to \tilde{H}_n(A \cap B) \to \tilde{H}_n(A) \oplus \tilde{H}_n(B) \to \tilde{H}_n(X) \to \tilde{H}_{n-1}(A \cap B) \to \cdots \to \tilde{H}_0(X) \to 0.$$

Theorem 1.2.2 (Mayer-Vietoris Sequence for Cohomology)**.** Suppose X is a topological space such that $X = \operatorname{int}(A) \cup \operatorname{int}(B)$, where A and B are subspaces of X and $\operatorname{int}(A)$ and $\operatorname{int}(B)$ denote the interiors of A and B, respectively. Then we obtain the following long exact sequence of singular cohomology groups of X with coefficients in the group G:

$$\cdots \to H^n(X; G) \to H^n(A; G) \oplus H^n(B; G) \to H^n(A \cap B; G) \to H^{n+1}(X; G) \to \cdots$$

Remark 1.2.3 (Mayer-Vietoris Sequence for de Rham Cohomology)**.** In Theorem 1.2.2, if X is a smooth manifold and $G = \mathbb{R}$, we recover the *Mayer-Vietoris sequence for de Rham cohomology.*

1.3 Relationship With the van Kampen Theorem & Homology Theories

Suppose X is a topological space such that $X = A \cup B$ for open, path-connected subspaces A and B of X that each contain the basepoint of X and whose intersection $A \cap B$ is path-connected. Then, by the Mayer-Vietoris sequence, we have an exact sequence

$$H_1(A \cap B) \to H_1(A) \oplus H_1(B) \to H_1(X) \to H_0(A \cap B) \to H_0(A) \oplus H_0(B)$$

Since the 0^{th} homology group of a path-connected space is isomorphic to $\mathbb{Z}$, this exact sequence simplifies to

$$H_1(A \cap B) \xrightarrow{\phi} H_1(A) \oplus H_1(B) \xrightarrow{\psi} H_1(X) \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$$

ψ is surjective by our path-connectedness assumptions, so by the First Isomorphism Theorem, we have $(H_1(A) \oplus H_1(B))/\ker(\psi) \cong H_1(X)$. Finally, by exactness, $\ker(\psi) = \operatorname{Im}(\phi)$, so we obtain $(H_1(A) \oplus H_1(B))/\operatorname{Im}(\phi) \cong H_1(X)$. Now, by our original assumptions, we also have by the van Kampen Theorem that there exists a homomorphism $\theta : \pi_1(A) *_{\pi_1(A \cap B)} \pi_1(B) \longrightarrow \pi_1(X)$, where $*_{\pi_1(A \cap B)}$ denotes an amalgamated free product of groups, which induces an isomorphism $\pi_1(X) \cong (\pi_1(A) *_{\pi_1(A \cap B)} \pi_1(B))/\ker(\theta)$. But, using that $H_1(Y) \cong \operatorname{Ab}\pi_1(Y)$ for a path-connected space Y[3], a union of path-connected subspaces of a

[2] The result used here is that a short exact sequence of chain complexes yields a wider long exact sequence of homology groups when passing to homology groups, and can be found in [5], pp. 116-117.

[3] This is the subject of Theorem 2A.1 in [5].

space is path-connected if they share at least one common point[4], and the abelianization of a free product of groups is isomorphic to the direct sum of their abelianizations, we recover precisely the previous isomorphism $(H_1(A) \oplus H_1(B))/\operatorname{Im}(\phi) \cong H_1(X)$. In this way, we see the analogue between the Mayer-Vietoris sequence and the van Kampen Theorem : the Mayer-Vietoris sequence yields exactly the abelianized statement of the van Kampen Theorem.

Recall that the Eilenberg-Steenrod axioms (detailed in [9] in terms of homology theories on pairs of spaces) provide certain properties satisfied by a given homology theory, including the excision axiom. Recall also the well-known example of a homology theory that is singular homology. In particular, the excision property satisfied by singular homology can be used to derive both the original and relative form of the Mayer-Vietoris sequence (see [3] for more detail). Even more, one may provide the axioms for homology theories in terms of homology theories on individual spaces, rather than pairs of spaces. In this case, one may replace the exactness axiom by one that involves the Mayer-Vietoris sequence [7]. Further, every homology theory of CW complexes has Mayer-Vietoris sequences. Finally, one can provide a different statement of the axioms for homology theories by applying the Mayer-Vietoris sequence[5].

2 Example Computations

In this section, we use the Mayer-Vietoris sequence and its version for reduced homology to calculate the homology groups or reduced homology groups of certain well-studied topological spaces.

Example 2.0.1 (Homology of the k-sphere)**.** Consider the n^{th} homology group $H_n(S^k)$ of the k-sphere $X = S^k$ $(n, k \geq 0)$. $H_n(S^k) \cong \mathbb{Z} \oplus \mathbb{Z}$ for $n = k = 0$, as the disjoint union of two points has two path-connected components. $H_n(S^k) \cong \mathbb{Z}$ for $n = 0, k > 0$ as S^k has one path-connected component for $k > 0$. Let $k \geq 1$, let A be the upper hemisphere of X, and let B be the lower hemisphere of X. Then, for $n \geq 1$, since $A \cap B = S^{k-1}$ and A and B are each contractible as retracts of a punctured sphere which is homeomorphic to $\mathbb{R}^n$ (and, hence, they each have vanishing homology in all positive degrees and $H_0(A) \cong H_0(B) \cong \mathbb{Z}$), we obtain via the Mayer-Vietoris sequence the exact sequence

$$0 \to H_{n+1}(S^k) \to H_n(S^{k-1}) \to 0,$$

giving that $H_{n+1}(S^k) \cong H_n(S^{k-1})$[6]. This isomorphism will allow us to complete our characterization of the homology groups of spheres once we compute homology in degree 1 for a general sphere of dimension $k \geq 1$. Via the Mayer-Vietoris sequence, we have the exact sequence

$$0 \to H_1(S^k) \to H_0(S^{k-1}) \to \mathbb{Z} \oplus \mathbb{Z} \to H_0(S^k) \to 0.$$

For $k > 1$, this yields the exact sequence

$$0 \to H_1(S^k) \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0,$$

showing that $H_1(S^k) = 0$[7], while for $k = 1$, we have $H_1(S^1) \cong \mathbb{Z}$[8]. Finally, it's easy to see that $H_1(S^0) \cong 0$. With this, we may fully characterize the homology groups of the k-sphere as follows :

$$H_n(S^k) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0, k > 0 \\ \mathbb{Z} & \text{if } n = k, k > 0 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } n = k = 0 \\ 0 & \text{otherwise} \end{cases}$$

[4]The relevant result here can be found in Proposition 16.14 (2) of [2].

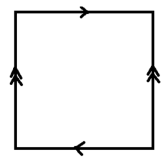
[5]This is discussed by Lennart Meier at https://mathoverflow.net/questions/97621/mayer-vietoris-implies-excision.

[6]Because the suspension of the n-sphere is the $(n+1)$-sphere, this agrees with the result that, for $n \geq 1$, $H_{n+1}(\Sigma X) \cong H_n(X)$, where X is a topological space. We will see in Example 2.04 that this coincides with the result based on reduced homology.

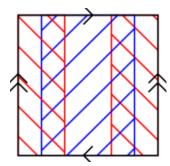
[7]This follows from exactness of the sequence $0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0$.

[8]Without prior knowledge of this fundamental isomorphism, one can see this by use of the Mayer-Vietoris sequence for reduced homology in this example.

Example 2.0.2 (Homology of the Klein bottle)**.** Consider the n^{th} homology group of the Klein bottle X. $H_0(X) \cong \mathbb{Z}$ because X is connected. Recall that we may construct $X = I^2/\sim$ by defining the equivalence relation $\sim$ on the unit square $I^2 = [0,1] \times [0,1]$, given by $(0,y) \sim (1,y)$ for $0 \leq y \leq 1$ and $(x,0) \sim (1-x,1)$ for $0 \leq x \leq 1$, and equipping $I^2/\sim$ with the quotient topology with respect to the canonical quotient map $g : I^2 \to I^2/\sim$. That is, X is constructed through the identification of points on the boundary of the unit square as follows :



Now, we may decompose X into two Möbius bands A and B intersecting along their boundaries. Using the aforementioned construction of X, this decomposition is shown below with A in blue and B in red.



Each of A and B deformation retract onto their core circles, meaning A and B are each homotopy equivalent to S^1. Further, $A \cap B$ is homotopy equivalent to S^1. Hence, because homology groups are invariant under homotopy equivalence, this gives $H_n(A)$, $H_n(B)$ and $H_n(A \cap B)$ are isomorphic to $H_n(S^1)$ for all $n \geq 0$ (of which we have a full calculation from the last example). With this, for $n > 2$, we arrive at an exact sequence

$$H_n(S^1) \oplus H_n(S^1) \to H_n(X) \to H_{n-1}(S^1) \to H_{n-1}(S^1) \oplus H_{n-1}(S^1)$$

by use of the Mayer-Vietoris sequence, which simplifies to the exact sequence

$$0 \to H_n(X) \to 0 \to 0,$$

showing that $H_n(X) \cong 0$ for $n > 2$. It remains to compute $H_1(X)$ and $H_2(X)$. By the Mayer-Vietoris sequence, the sequence

$$H_2(S^1) \oplus H_2(S^1) \to H_2(X) \to H_1(S^1) \to H_1(S^1) \oplus H_1(S^1) \to H_1(X) \to 0$$

is exact, which simplifies to the exact sequence

$$0 \to H_2(X) \to \mathbb{Z} \xrightarrow{f} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{g} H_1(X) \xrightarrow{h} 0.$$

The map f is given by $f : \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} : 1 \longmapsto (2, -2)$ as, for each Möbius band, its boundary wraps around its core circle twice. That is, f is a degree 2 map that takes into account orientations. f is injective, so we have $H_2(X) \cong 0$ by the First Isomorphism Theorem. Further, with $\text{Im}(f) = 2\mathbb{Z}(1,-1) = \ker(g)$ and h the zero map, g is surjective and $(\mathbb{Z} \oplus \mathbb{Z})/2\mathbb{Z}(1,-1) \cong H_1(X)$ by the First Isomorphism Theorem. Finally, as $\{(1,0),(1,-1)\}$ is a basis for $\mathbb{Z} \oplus \mathbb{Z}$, we obtain by direct calculation that $H_1(X) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. In summary, the homology of the Klein bottle is shown below.

$$H_n(X) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } n = 1 \\ 0 & \text{if } n \geq 2 \end{cases}$$

Example 2.0.3 (Reduced homology of a wedge sum)**.** Consider the n^{th} reduced homology group of the wedge product $X \vee Y$ of two spaces X and Y, where we suppose that we have open neighborhoods $U \subset X$ and $V \subset Y$ and each of U and V deformation retract down to the basepoints of X and Y that are identified when constructing $X \vee Y$. Let $A = X \cup V$ and let $B = Y \cup U$. Then, by the Mayer-Vietoris sequence for reduced homology, since $A \cap B = U \cap V$, we have for all n the exact sequence

$$\tilde{H}_n(U \cap V) \to \tilde{H}_n(X \cup V) \oplus \tilde{H}_n(Y \cup U) \to \tilde{H}_n(X \vee Y) \to \tilde{H}_{n-1}(U \cap V).$$

But $U \cap V$ deformation retracts down to a point (hence its reduced homology groups all vanish), $X \cup V$ deformation retracts down to X, and $Y \cup U$ deformation retracts down to Y, so using that homology is invariant under deformation retraction, the exact sequence simplifies to

$$0 \to \tilde{H}_n(X) \oplus \tilde{H}_n(Y) \to \tilde{H}_n(X \vee Y) \to 0.$$

Thus, $\tilde{H}_n(X) \oplus \tilde{H}_n(Y) \cong \tilde{H}_n(X \vee Y)$ for all n. [9]

Example 2.0.4 (Reduced homology of the suspension of a space)**.** Consider the n^{th} reduced homology group of the suspension ΣX of a space X. In constructing ΣX, each of $X \times \{0\}$ and $X \times \{1\}$ are collapsed down to points p and q, respectively. Let $A = \Sigma X - \{p\}$ and let $B = \Sigma X - \{q\}$. Each of A and B are homeomorphic to the cone space $CX = (X \times I)/(X \times \{0\})$. By the Mayer-Vietoris sequence for reduced homology, since $A \cap B = X \times (0, 1)$, we obtain the exact sequence

$$\tilde{H}_n(\Sigma X - \{p\}) \oplus \tilde{H}_n(\Sigma X - \{q\}) \to \tilde{H}_n(\Sigma X) \to \tilde{H}_{n-1}(X \times (0,1)) \to \tilde{H}_{n-1}(\Sigma X - \{p\}) \oplus \tilde{H}_{n-1}(\Sigma X - \{q\})$$

for all n. But, by using that homology is invariant under homeomorphism, that CX is contractible[10] (hence, its reduced homology vanishes in all degrees), that $X \times (0, 1)$ deformation retracts down to X, and that homology is invariant under deformation retraction, this sequence simplifies to

$$0 \to \tilde{H}_n(\Sigma X) \to \tilde{H}_{n-1}(X) \to 0$$

Thus, we obtain that the n^{th} reduced homology group of the suspension of a space X is isomorphic to the $(n-1)^{\text{st}}$ homology group of X for all n. [11]

3 The Mayer-Vietoris Spectral Sequence

In this section, we describe the *Mayer-Vietoris spectral sequence*, from which we can recover the Mayer-Vietoris sequence. The fundamental idea is to now consider an open cover of a space X by any number of open subsets of X, and utilize the spectral sequence to compute the homology of X.

Suppose that X is a topological space and $\mathcal{U} = \{U_1, U_2\}$ is an open cover of X. Then, following the same ideas we've used previously in our proof of the Mayer-Vietoris sequence, we have a short exact sequence of chain complexes

$$0 \to S_\bullet(U_0 \cap U_1) \to S_\bullet(U_0) \oplus S_\bullet(U_1) \to S_\bullet^{\mathcal{U}}(X) \to 0,$$

where $S_q(Y) := \mathbb{Z}[\text{map}(\Delta^q, Y)]$, $S_q^{\mathcal{U}} = \mathbb{Z}[\{\sigma : \Delta^2 \to X \mid \exists U \in \mathcal{U} \text{ s.t. } \text{Im}(\sigma) \subseteq U\}]$, the map from $S_\bullet(U_0 \cap U_1)$ to $S_\bullet(U_0) \oplus S_\bullet(U_1)$ maps an element x to $(-x, x)$, and the map from $S_\bullet(U_0) \oplus S_\bullet(U_1)$ to $S_\bullet^{\mathcal{U}}(X)$ maps an element (y, z) to $y + z$. Applying the long exact sequence in homology to this short exact sequence yields a long exact sequence of singular homology groups precisely like the Mayer-Vietoris sequence.

Definition 3.0.1. Suppose that X is a topological space and $\mathcal{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ is an open cover of X. Then, we can construct the corresponding abstract simplicial complex given by $K_{\mathcal{U}} = (V_{K_{\mathcal{U}}}, S_{K_{\mathcal{U}}})$, where the vertices $V_{K_{\mathcal{U}}} = \{U_\lambda\}_{\lambda \in \Lambda}$ are the open sets in the cover and $S_{K_{\mathcal{U}}} = \{\overline{\Gamma} := \{U_\gamma\}_{\gamma \in \Gamma} : U_\Gamma = \cap_{\gamma \in \Gamma} U_\gamma \neq \emptyset, \emptyset \neq \Gamma \subseteq \Lambda \text{ finite}\}$ gives the simplices. Further, there exists an *extended ordered Cech complex*, which is the long exact sequence of chain complexes

$$\cdots \xrightarrow{\delta_{2,2}} \oplus_{\Gamma \in K_{\mathcal{U}}^{(1)}} S_q(U_\Gamma) \xrightarrow{\delta_{1,2}} \oplus_{\gamma_0} S_q(U_{\gamma_0}) \to S_q^U(X) \to 0.$$

[9] This isomorphism holds for the ordinary unreduced homology groups as well, except for in degree zero (hence the advantage to using reduced homology here).

[10] This is due to the homotopy $h_t(x, s) = (x, (1 - t)s)$, continuously shrinking CX down to its vertex point.

[11] The same isomorphism holds for $n \geq 1$ in the case that we use unreduced homology here.

The *Mayer-Vietoris double complex* (shown below) (M, d^ζ, d^η) is then given by $M_{p,q} := \oplus_{\Gamma\in K_{\mathcal{U}}^{(p)}} S_q(U_\Gamma)$ with vertical differentials derived from direct sums of the corresponding differentials between singular homology groups and horizontal differentials derived from the corresponding differentials in the appropriate Cech complex.

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 & \oplus_{\Gamma\in K_{\mathcal{U}}^{(0)}} S_2(U_\Gamma) \arrow[l] \arrow[d] & \oplus_{\Gamma\in K_{\mathcal{U}}^{(1)}} S_2(U_\Gamma)) \arrow[l] \arrow[d] & \oplus_{\Gamma\in K_{\mathcal{U}}^{(2)}} S_2(U_\Gamma) \arrow[l] \arrow[d] & \cdots \arrow[l] \\
0 & \oplus_{\Gamma\in K_{\mathcal{U}}^{(0)}} S_1(U_\Gamma) \arrow[l] \arrow[d] & \oplus_{\Gamma\in K_{\mathcal{U}}^{(1)}} S_1(U_\Gamma)) \arrow[l] \arrow[d] & \oplus_{\Gamma\in K_{\mathcal{U}}^{(2)}} S_1(U_\Gamma) \arrow[l] \arrow[d] & \cdots \arrow[l] \\
0 & \oplus_{\Gamma\in K_{\mathcal{U}}^{(0)}} S_0(U_\Gamma) \arrow[l] \arrow[d] & \oplus_{\Gamma\in K_{\mathcal{U}}^{(1)}} S_0(U_\Gamma)) \arrow[l] \arrow[d] & \oplus_{\Gamma\in K_{\mathcal{U}}^{(2)}} S_0(U_\Gamma) \arrow[l] \arrow[d] & \cdots \arrow[l] \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Finally, arising from this double complex is a (convergent) spectral sequence, the *Mayer-Vietoris spectral sequence* $E^2_{p,q} = H_p(C_\bullet(K_{\mathcal{U}}, F_q)) \Rightarrow H_{p+q}(X)$. The differential d^r is defined by $d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$. By performing spectral sequence computations at the first and second page of this spectral sequence, we can obtain two respective short exact sequences, and by splicing these two sequences, we can recover the Mayer-Vietoris sequence from this spectral sequence.

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